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ADVANCED ALGEBRA

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PREFACE

THIS *Advanced Algebra* aims to present the vital topics of the course in such a way as to bring them within the comprehension of the pupil, at the same time giving him a working knowledge of the subject. It is designed for students who have had from one to two years of algebra.

Nothing has been slighted that seems necessary for a thorough understanding of the subject, and we have taken pains to suggest methods which should prove illuminating in the later work of the pupil. At the same time we have included no theoretical considerations merely for their own sakes, and have chosen as far as possible such methods of proof and such forms of expression as experience has taught us are most easily understood by the pupil. The syllabi of the College Entrance Board, of the Regents of the State of New York, and of practically all states have been covered.

The emphasis on review which has been a feature of our more elementary texts has been carefully maintained in this book. Here it takes the form of cumulative reviews at the end of each chapter with occasional shorter reviews elsewhere. The book concludes, as in the case of our earlier works, with extensive general reviews covering all topics of the course.

For valuable suggestions we acknowledge our indebtedness to colleagues in our respective schools, and to the many teachers who are using our books.

EDWARD I. EDGERTON
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ADVANCED ALGEBRA

CHAPTER I

FUNDAMENTAL LAWS AND OPERATIONS

WE assume that students, who study advanced algebra, are already familiar with the technique of the subject. However, it seems desirable to present some discussion of the underlying principles of the subject, as well as a short review of the more difficult portions of elementary algebra with somewhat more difficult examples than are found in the earlier courses.

Fundamental Laws

Operations upon algebraic quantities are performed according to certain mathematical laws, which you no doubt recognized and applied intuitively, but which you should now be able to state. The first of these laws is called

1. The Commutative Law. — As applied to addition this means that the order of the elements to be added does not affect the sum. It is a matter of common knowledge that $2 + 3 + 4 = 2 + 4 + 3 = 3 + 4 + 2$, etc. Since $5 + 7 - 3$ may be regarded as $5 + 7 + (-3)$, it follows that these quantities also may be taken in any order. That is, *the operations of addition and subtraction may be performed in any order.*

As applied to multiplication the commutative law means that the product is not affected by the order of the factors. For example, $5 \cdot 4 \cdot 7 = 5 \cdot 7 \cdot 4 = 4 \cdot 7 \cdot 5$, etc. Also abc

$= bac = acb$ etc. Hence, *in a series of operations involving only multiplication, the multiplication may be performed in any order.*

As applied to division we note that $12 \div 2 \div 3 = 6 \div 3 = 2$, while $12 \div 3 \div 2 = 4 \div 2 = 2$, or $a \div b \div c = a \div c \div b$. Hence, *in a series of operations involving only division, the divisors may be written in any order.*

Let the pupil state a law concerning the order of operations in a series involving addition, subtraction, multiplication, and division. See authors' First Course in Algebra, page 6.

2. The Associative Law. — The statement $5 + (3 + 1)$ means that we are to add 3 and 1 making 4 and then add this result to 5, producing 9. We might equally well have written $(5 + 3) + 1$, thus associating 5 and 3, or $3 + (5 + 1)$, thus treating $5 + 1$ as a single quantity. Hence, the sum of b and c added to a is the same as the sum of a and b added to c , or the sum of a and c added to b .

A similar law holds for multiplication. Thus $a(bc) = (ab)c = (ac)b = abc$. That is, a times the product of b and c is equal to c times the product of a and b , etc. Obviously this principle may be extended to include any number of factors.

3. The Distributive Law. — In the product $3(4 + 5)$ we have $3 \cdot 9 = 27$. We obtain the same result by multiplying 4 and 5 in succession by 3 and adding the results. Thus $3(4 + 5) = 3 \cdot 4 + 3 \cdot 5 = 12 + 15 = 27$. Hence, $a(b + c) = ab + ac$, etc. Hence *the product of an algebraic sum by a single multiplier is equal to the sum of the products obtained by multiplying each term in the sum by that multiplier.*

A similar law holds for division. For, $(15 + 10) \div 5 = \frac{25}{5} = 5$. Also $\frac{15}{5} + \frac{10}{5} = 3 + 2 = 5$. That is,

$$\frac{(a + b + c + \dots)}{m} = \frac{a}{m} + \frac{b}{m} + \frac{c}{m} + \dots$$

Hence, *the quotient of an algebraic sum by a single divisor is equal to the corresponding algebraic sum of the quotients of each term in the sum by that divisor.*

Elementary Processes

4. Factoring. — You are advised to consider the following cases of factoring; (a) removal of a common monomial factor, (b) grouping to show a common binomial or trinomial factor, (c) quadratic trinomials, (d) difference of two squares, (e) expressions reducible to the difference of squares, (f) sum or difference of odd powers, (g) factor theorem. If necessary, refer to the authors' Second Course in Algebra for a detailed discussion of these cases.

Exercises

1. Prepare type forms to illustrate each of the above cases of factoring.

2. Prepare and factor two examples of each of the above type forms. Factor each of the following, being sure to obtain prime factors.

3. $x^{12} - 16 y^{16} z^8$.
4. $6(2 m^2 - m)^2 + 10 m^2 - 5 m - 6$.
5. $4 a^2 - 19 a + 15$.
6. $y^3 - 27 - 19(y - 3)$.
7. $2 a x^2 + 3 b m y - 3 a m x - 2 b x y$.
8. $33 x + 71 x^2 - 14 x^3$.
9. $4(x^2 - 9) - 2 x^2 - 5 x + 3$.
10. $x^8 + 64$.
11. $143 x^2 + 144 x y - 119 y^2$.
12. $x^3 - 2 x^2 - 9 x + 18$.
13. $32 a^5 - b^{10}$.
14. $1331 x^{27} - 343 y^{18}$.
15. $(x^2 - y^2 - z^2)^2 - 4 y^2 z^2$.

16. $(x^2 - 1)^2 + 3(2x + 1)(x + 1)^2$.
 17. $x^2 - y^2 - 16z^2 + 4m^2 - 4mx + 8yz$.
 18. $m^4 + n^4 - 119m^2n^2$. 20. $x^{3n} - 27y^{3m}$.
 19. $1 - (x - 1)^3$. 21. $x^3 + 18x^2 + 108x + 216$.
 22. $(m + n)(a - b)^2 - (m^2 - n^2)(a - b)$.
 23. $a^n - b^n$ if n is odd. 25. $y^3 - 4y^2 + 3$.
 24. $a^{2n} - b^{2n}$ (3 factors).* 26. $x^4 - 3x^2y^2 + y^4$.
 27. $(y^2 + 2y - 1)^2 - (y^2 - 2y + 1)^2$.
 28. $H^3 + K^3 + 3HK(H + K)$.
 29. $2(x^3 - 1) + (x - 1)(11x + 2)$.
 30. $\frac{1}{4}x^{12} - 16$. 31. $.008x^3 - .027y^3$.

5. Operations with Fractions. — Skill in the manipulation of fractional expressions is valuable. The fraction is much used in algebra because of our habit of representing division by the fractional form.

Exercises

Reduce to lowest terms :

1. $\frac{3x^5 - 3x^3y^2 + 3x^4y - 3x^2y^3}{9x^4y - 9x^3y^2 - 9x^2y^3 + 9xy^4}$.
 2. $\frac{x^3 - 2x^2 + 1}{x^3 - 2x - 1}$. 3. $\frac{48x^2 - 64xy + 21y^2}{36x^2 + 17xy - 33y^2}$.
 4. $\frac{x^3 - 8}{x^3z + 2x^2z + 4xz}$.

Collect :

5. $\frac{x^2 - 1}{x^2 - x - 6} - \frac{2x + 1}{x^2 + x - 1} - \frac{1 - x^2}{x^2 - 4x + 3}$.
 6. $\frac{2c + a}{x + a} - \frac{4cx - 2a^2}{x^2 - a^2} - \frac{2c - x}{a - x}$.

* 2 irrational.

$$7. \frac{x^2 - x - 6}{x^2 + 5x + 6} - \frac{15x - x^2}{9x - x^3} - \frac{2x^2 + 8x + 6}{2x^2 - 4x - 6}.$$

(Reduce each fraction to lowest terms before adding.)

$$8. \frac{1}{9x^2 + 9x + 2} - \frac{1}{9x^2 - 9x + 2} + \frac{1}{4 - 9x^2} - \frac{1}{1 - 9x^2}.$$

$$9. \frac{3}{x^2 - 3x + 2} + \frac{1}{5x - x^2 - 6} + \frac{2}{4x - 3 - x^2}.$$

$$10. \frac{a}{(b - c)(c - a)} + \frac{b}{(c - a)(a - b)} - \frac{c}{(a - b)(b - c)}.$$

$$11. \frac{a^3}{(a - b)(a - c)} + \frac{b^3}{(b - c)(b - a)} + \frac{c^3}{(c - a)(c - b)}.$$

$$12. \frac{1}{4x} - \frac{1}{2} \left(\frac{1}{x} \left[\frac{\frac{1}{2}}{2x - 1} + \frac{x - 1}{4x^2 - 1} \right] - \frac{\frac{1}{2}}{2x + 1} \right).$$

13. Show that

$$16 + \left\{ \frac{x + a}{x - a} + \frac{x - a}{x + a} - 2 \frac{x^2 - a^2}{x^2 + a^2} \right\}^2 = 16 \left(\frac{x^4 + a^4}{x^4 - a^4} \right)^2.$$

Simplify :

$$14. \frac{n}{n^2 - 4} \div \frac{n}{3n^2 + 5n + 2} \left(3n - \frac{2}{n} - 5 \right).$$

$$15. \frac{1 - x^3}{1 + x^2} \div \left(\frac{1 - x^2 + x^4}{x^3 + 1} \right) \div \left(1 - \frac{2x^6}{x^6 + 1} \right).$$

$$16. \frac{\frac{b^2}{a^2} + 1 + \frac{a^2}{b^2}}{\frac{b}{a} + 1 + \frac{a}{b}}.$$

$$17. \left(\frac{3}{a} - \frac{a}{3} \right) \left(\frac{9}{a^2} + 1 \right) \left[\frac{\frac{9}{a^2} + \frac{a^2}{9}}{\frac{9}{a^2} - \frac{a^2}{9}} - 1 \right].$$

$$18. \frac{x^4 - 5x^2 + 4}{x^2 - 1} \cdot \frac{1 + \frac{1}{x}}{2 + \frac{4}{x}} \div \frac{x - \frac{1}{x}}{\frac{1}{x} - \frac{1}{x^2}}.$$

$$19. 2 + \frac{3}{4 + \frac{5}{6 + \frac{1}{x}}}.$$

$$20. \frac{2}{3} \left\{ \frac{5}{2a} - \frac{1}{4} \left[\frac{2}{3} - \frac{1}{4} (2a - 6[2a + 1]) \right] \right\} - \frac{2a}{3} + \frac{14}{3}.$$

21. In the expression $\frac{u^2 - v}{1 - uv}$, substitute $u = \frac{x^2 - y}{1 - xy}$, $v = \frac{y^2 - x}{1 - xy}$; and simplify the result as far as possible.

22. Simplify $\frac{x^{\frac{1}{2}} + 1}{x^{\frac{1}{2}} - 1} - \frac{x^{\frac{1}{2}} - 1}{x^{\frac{1}{2}} + 1} - \frac{4x^0}{x - 1}$ and check by letting $x = 4$.

23. Simplify $\left(1 - \frac{x^2}{y^2}\right) \left(1 - \frac{xy - x^2}{y^2}\right) \frac{y^4}{y^3 + x^3} \cdot \frac{y - x}{y^2 + x^2}$ and find the numerical result when $x = .3$ and $y = 4.5$ correct to hundredths.

24. Simplify $\left(\frac{x - y}{x + y} - \frac{x^3 - y^3}{x^3 + y^3}\right) \left(\frac{x - y}{x + y} + \frac{x^2 + y^2}{x^2 - y^2}\right)$, and check with $x = 3$, and $y = 2$.

6. Fractional Equations. — Linear equations containing fractions may be solved by the following

Rule. — *Simplify the equation by reduction of complex fractions, removal of parentheses, etc. Remove fractions from the equation by multiplying both members by the lowest common multiple of the denominators. Solve the resulting simple equation. Check.*

Illustrative example I :

Solve :

$$\frac{2x - .2}{4x + .5} = \frac{.2x - 1.6}{.4} - \frac{x}{2}.$$

Solution: Reducing the first two fractions to simple fractions by multiplying numerator and denominator in each case by 10, the lowest common multiple of the decimal denominators, we obtain

$$\frac{20x - 2}{40x + 5} = \frac{2x - 16}{4} - \frac{x}{2}.$$

Multiply both members by $4(40x + 5)$.

$$80x - 8 = \cancel{80x^2} - 630x - 80 - \cancel{80x^2} - 10x.$$

$$720x = -72.$$

$$x = -.1.$$

Check in the original equation.

We call your attention to the following special methods.

Illustrative example II :

Solve :

$$\frac{2x + 3}{5} + \frac{x + 24}{10} - \frac{5x}{6} = \frac{22x + 12}{13x - 6} - \frac{2x - 6}{6}.$$

When all the denominators are monomials except one it is sometimes best to clear of fractions in two stages of work, multiplying first by the L.C.M. of the monomial denominators, and afterwards by the polynomial denominator.

Thus multiplying by 30,

$$12x + 18 + 3x + 72 - 25x = \frac{30(22x + 12)}{13x - 6} - 10x + 30.$$

Whence

$$60 = \frac{30(22x + 12)}{13x - 6}.$$

Or

$$1 = \frac{11x + 6}{13x - 6}.$$

Multiplying by $13x - 6$,

$$13x - 6 = 11x + 6.$$

$$2x = 12.$$

$$x = 6.$$

Illustrative example III :

Solve
$$\frac{1}{x-3} - \frac{1}{x-4} = \frac{1}{x-5} - \frac{1}{x-6}.$$

If this were solved in the usual way, we should multiply by $(x-3)(x-4)(x-5)(x-6)$ and the solution would become quite difficult. Note the peculiar relation of the denominators. We shall add the fractions in each member of the equation, and obtain

$$\begin{aligned} \text{Hence } \frac{-1}{(x-3)(x-4)} &= \frac{-1}{(x-5)(x-6)}. \\ (x-5)(x-6) &= (x-3)(x-4). \\ x^2 - 11x + 30 &= x^2 - 7x + 12. \\ 4x &= 18. \\ x &= \frac{9}{2}. \end{aligned}$$

Exercises

Solve each of the following equations :

$$1. \frac{x+3}{3(x-3)} = \frac{x^2+6}{6(9-x^2)} + \frac{x-3}{2x+6}.$$

$$2. \frac{6(x+1)}{2x^2+5x+3} - \frac{2}{x+1} = -\frac{1+2x}{1+x-2x^2}.$$

$$3. \frac{3}{4}(x-2) + \frac{3}{5}(x+5) - \frac{1}{7}(2x+1) = \frac{3}{8}(3x+5) - 1\frac{1}{8}.$$

$$4. \frac{1.3x - .05}{4} - \frac{3.035 - .8x}{.24} = \frac{.5x + .04}{.06}.$$

$$5. \frac{25 - \frac{x}{3}}{x+1} - 5 = \frac{23}{x+1} - \frac{16x + 41\frac{1}{2}}{3x+2}.$$

$$6. \frac{4(x+3)}{9} - \frac{8x+37}{18} = \frac{29-7x}{5x-12}.$$

$$7. \frac{x}{6} - \frac{2-3x}{9} + \frac{3-2x}{4} = \frac{6x-1}{15-7x}.$$

$$8. \frac{x+1}{x-1} - \frac{x}{x-2} = \frac{x-9}{x-7} - \frac{x-8}{x-6}.$$

$$9. \frac{x+2}{x} - \frac{x+3}{x+1} = \frac{x-6}{x-4} - \frac{x-7}{x-5}.$$

$$10. \frac{y-7}{y-9} - \frac{y-9}{y-11} = \frac{y-13}{y-15} - \frac{y-15}{y-17}.$$

$$11. \left(\frac{x+c}{2}\right)^2 = \frac{x(x-c)}{4} - \frac{2c}{3}\left(x - \frac{c}{2}\right).$$

$$12. \frac{x+ab}{a+b} - \frac{x-ab}{a-b} = \frac{2ab}{a+b}.$$

$$13. \frac{3m}{5}\left(m + \frac{x}{m}\right) + \left(m - \frac{x}{m}\right)\frac{2m}{3} = 0.$$

$$14. \frac{\frac{x-1}{2}}{1 + \frac{2x}{3}} = \frac{\frac{3x-2}{2}}{\frac{5x}{2}}.$$

$$15. \left(1 + \frac{x}{a}\right)\left(\frac{1}{a} - x\right) = (x-a)\left(1 - \frac{x}{a}\right).$$

$$16. \frac{m(3a^2 - 2b^2)}{a+b} = \frac{2a^2 - b^2}{ax+bx} + \frac{a-b}{x}.$$

$$17. \frac{\frac{3}{4} - x}{\frac{1}{4}(1+x)} + \frac{\frac{7}{3}}{1 - \frac{1}{x^2}} = -\left(\frac{1}{3} + \frac{x - \frac{2}{3}}{\frac{3}{4}(x-1)}\right).$$

$$18. \frac{1}{2}\{x - \frac{2}{3}(2x - \frac{1}{4}[3x + (x-3) + \frac{2}{3}] - \frac{1}{2}) + x - 2\} = 0.$$

$$19. \frac{x - \frac{5}{3}}{\frac{5}{3}(x-1)} + \frac{\frac{7}{3} - x}{\frac{7}{3}(1+x)} = \frac{1}{35\left(1 - \frac{1}{x^2}\right)} + \frac{1}{7}.$$

$$20. \frac{3}{1-3x} + \frac{5}{1-5x} = \frac{4}{1-2x}.$$

$$21. \frac{x^2 + a^2}{4x^2 - a^2} - \frac{x}{2x + a} = -\frac{1}{4}.$$

$$22. \frac{1}{a^2} - \frac{1 + \frac{1}{a}}{\frac{a}{x}\left(1 - \frac{1}{a^2}\right)} = -\frac{1}{a}.$$

$$23. \frac{3x-1}{x+3} - \frac{2x+3}{1-x} = \frac{5x^2-2x-1}{x^2+2x-3}.$$

$$24. \frac{5y-.4}{.3} + \frac{1.3y-.05}{2} = \frac{13.95-8y}{1.2}.$$

$$25. \frac{2.4x-.12}{2.8} + \frac{4.6x-3.6}{4} = \frac{.64x-.048}{.7}.$$

$$26. a(x-b) + b(x-c) + c(x-a) = 0.$$

$$27. \frac{x-a}{2bc} + \frac{x-b}{2ac} + \frac{x-c}{2ab} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

$$28. \frac{1-1.4x}{.2+x} = \frac{.7(x-1)}{.1-.5x}.$$

$$29. \frac{5x-8}{x-2} + \frac{6x-44}{x-7} = \frac{x-8}{x-6} + \frac{10x-8}{x-1}.$$

$$30. \frac{b - \frac{1}{y}}{b + \frac{1}{y}} - \frac{1}{y} = \frac{y - \frac{1}{b}}{y + \frac{1}{b}} - \frac{1}{b}.$$

7. Laws of Exponents. — The laws governing the use of exponents may be thus expressed:

(a) For multiplication $a^m \cdot a^n = a^{m+n}.$

(b) For division $\frac{a^m}{a^n} = a^{m-n}.$

(c) For involution (raising to powers) . . . $(a^m)^n = a^{mn}.$

- (d) For evolution (extracting roots). $\sqrt[n]{a^m} = (\sqrt[n]{a})^m = a^{\frac{m}{n}}$.
- (e) For powers of products $(abc)^n = a^n b^n c^n$.
- (f) For powers of quotients $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

The meanings of the various types of exponents are indicated by the following forms.

1. When the exponent is a positive integer,

$$a^n = a \cdot a \cdot a \cdots \text{to } n \text{ factors.}$$

2. When the exponent is 0,

$$a^0 = 1.$$

3. When the exponent is a positive fraction,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

4. When the exponent is negative,

$$a^{-n} = \frac{1}{a^n}.$$

Let the pupil show that statement (3) follows as a result of law (d), and that statements (2) and (4) follow as a result of law (b). See authors' Second Course, Chapter 6.

From statement (4) above we obtain the following

Principle: *Inverting a factor in a product is equivalent to changing the sign of the exponent of that factor. Hence if a factor is both inverted and has the sign of its exponent changed, the value of the product is unchanged.*

Examples:

$$\left(\frac{a}{b}\right)^{-2} = \left(\frac{b}{a}\right)^2; \frac{a^{-2}b}{c^2} = \frac{b}{a^2c^2}; \frac{m^{-2}n^2}{a^{-3}b^{-2}} = \frac{a^3b^2n^2}{m^2}.$$

8. Radicals. — A *radical* is an indicated root.

The *index* of a radical is the small figure within the $\sqrt{}$ of the radical sign which denotes the kind of root to be taken.

The *order* of a radical is indicated by its index.

The *radicand* is the quantity under the radical sign.

The following principles govern the simplification of radicals.

1. Extract the indicated root, if it can be done exactly.

Examples: $\sqrt{9} = 3$; $-\sqrt{\frac{4}{9}} = -\frac{2}{3}$; $\sqrt[3]{-27} = -3$.

2. Reduce the radical to the lowest possible order.

Examples: $\sqrt[4]{4} = \sqrt{2}$; $\sqrt[6]{9} = \sqrt[3]{3}$; $\sqrt[12]{49} = \sqrt[6]{7}$.

3. Make the radicand as small as possible.

Examples:

$\sqrt{40} = \sqrt{4 \cdot 10} = 2\sqrt{10}$; $\sqrt[3]{-72} = \sqrt[3]{9(-8)} = -2\sqrt[3]{9}$.

4. Make the radicand integral.

Examples:

$$\sqrt{\frac{1}{2}} = \sqrt{\frac{1}{2} \cdot \frac{2}{2}} = \sqrt{\frac{1}{4} \cdot 2} = \frac{1}{2}\sqrt{2}.$$

$$\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{2}{3} \cdot \frac{9}{9}} = \sqrt[3]{\frac{18}{27}} = \frac{1}{3}\sqrt[3]{18}.$$

5. Make the denominators of all fractions rational.

Examples:

$$\frac{3}{2\sqrt{3}} = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}.$$

$$\frac{2}{\sqrt[3]{6}} = \frac{2}{\sqrt[3]{6}} \cdot \frac{\sqrt[3]{36}}{\sqrt[3]{36}} = \frac{2\sqrt[3]{36}}{6} = \frac{\sqrt[3]{36}}{3}.$$

$$\begin{aligned} \frac{2 + 2\sqrt{3}}{3 - 2\sqrt{3}} &= \frac{2 + 2\sqrt{3}}{3 - 2\sqrt{3}} \cdot \frac{3 + 2\sqrt{3}}{3 + 2\sqrt{3}} = \frac{18 + 10\sqrt{3}}{-3} \\ &= -\frac{18 + 10\sqrt{3}}{3}. \end{aligned}$$

Exercises

Simplify:

1. $4^{\frac{3}{2}} + 16^{\frac{1}{4}} + 4^{-2} - 3^{-3} + 2^0$.
2. $a^{\frac{1}{3}} \cdot \sqrt[3]{c} \cdot \sqrt{b^3} \cdot b^{-\frac{1}{2}} \cdot \sqrt[3]{c^{-2}}$.
3. $(\frac{1}{32})^{\frac{6}{5}} \cdot (\frac{8}{125})^{-\frac{5}{3}}$.
4. $\frac{4^{-\frac{1}{2}} \cdot 9^{-\frac{3}{2}} \cdot 16^{\frac{1}{4}}}{16^{-1} \cdot 8^{-\frac{2}{3}}}$.
5. $\left[\sqrt[4]{\left(\frac{a^2}{4}\right)^{-2}} \right]^{-1}$.
6. $\sqrt[4]{[\sqrt[3]{(81^{-6})}]^2}$.
7. $\left(\frac{a^{n-1}}{a^n}\right)^{-n} \div \left(\frac{a^{-n}}{a^{n+1}}\right)^{1-n}$.
8. $\left(\frac{a^{-1}b^3}{a^3b^{-4}}\right)^{-3} \div \left(\frac{a^2b^{-2}}{a^4b}\right)^4$.
9. $\frac{\sqrt[3]{x^{-1}y} \sqrt{z^3}}{\sqrt{y^{-1}z^2} \sqrt[3]{x^2}}$.
10. $\left[\frac{a^{-2}b^2 \sqrt[3]{c^{-2}d^2a^2}}{a^2b^{\frac{1}{2}} \sqrt{a^3c}} \right]^{-\frac{1}{2}}$.
11. $\left(\frac{a^{-2}b}{c^2a^3}\right)^{\frac{1}{2}} \cdot \left(\frac{ac}{b^{-3}}\right)^{\frac{1}{3}} \cdot \left(\frac{-c^3d^2}{a^{-3}}\right)^{\frac{1}{3}}$.
12. $\left[\frac{x^{\frac{1}{2}} \sqrt[3]{y}}{\sqrt[4]{bx^2}} \cdot \frac{\sqrt{xy^{-3}}}{x^{-2}b^2y^{-1}} \cdot \frac{b^{\frac{1}{2}}x^{-2}}{\sqrt{xy^{-1}}} \right]^{-2}$.

 Solve for x :

13. $x^{\frac{2}{3}} = 9$.
14. $x^{-\frac{1}{2}} = -9$.
15. $x^{\frac{3}{4}} = -8$.
16. $2x^{-\frac{4}{3}} = 162$.

Simplify:

17. $\sqrt{6} \cdot \sqrt[3]{4} \cdot \sqrt[4]{2}$.
18. $3\sqrt{2} \div -4\sqrt{3}$.
19. $-12\sqrt[3]{6} \div 2\sqrt[3]{10}$.
20. $3\sqrt[3]{4} \div 2\sqrt{3}$.
21. $\frac{5 + \sqrt{5}}{5 - \sqrt{5}}$.
22. $\frac{2\sqrt{3} - 4\sqrt{2}}{\sqrt{3} + 2\sqrt{2}}$.
23. $\frac{5\sqrt{6} - 3\sqrt{2}}{5\sqrt{6} + 3\sqrt{2}}$.
24. $\frac{5\sqrt{3} - 1}{2\sqrt{3} - 1}$.
25. $\frac{3\sqrt{a} + 2\sqrt{b}}{3\sqrt{a} - 2\sqrt{b}}$.

Multiply:

$$26. (\sqrt{5} + 2\sqrt{2} - \sqrt{3})(2\sqrt{5} + \sqrt{2} - 4\sqrt{3}).$$

$$27. (3\sqrt{5} - 2\sqrt{2} + 1)(3\sqrt{5} - 2\sqrt{2} - 1).$$

$$28. (2\sqrt{3} + 3\sqrt{2} - \sqrt{6} + 1)(3\sqrt{2} - 4\sqrt{3}).$$

$$29. 2x^{\frac{3}{2}} + 3x - 6x^{\frac{1}{2}} + 1 \text{ by } x^{\frac{1}{2}} - x^{\frac{1}{4}} - 1.$$

$$30. 3x^{\frac{1}{2}} + 5 - 2x^{-\frac{1}{2}} + 3x^{-1} \text{ by } 2 - x^{\frac{1}{2}} - 2x^{-1}.$$

$$31. a^2 + a^{\frac{3}{2}}b^{-1} - 3ab^{-2} + 4a^{\frac{1}{2}}b^{-\frac{3}{2}} \text{ by } a - 2a^{\frac{1}{2}}b^{-1} + 3b^{-2}.$$

Divide:

$$32. a^{\frac{1}{2}} + a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}} \text{ by } a^{\frac{1}{4}} - a^{\frac{1}{8}}b^{\frac{1}{8}} + b^{\frac{1}{4}}.$$

$$33. 5 - 2x^{\frac{1}{8}} + x^{-\frac{1}{2}} - x^{-\frac{1}{8}} \text{ by } 2x^{\frac{1}{8}} + 1.$$

$$34. x^{\frac{3}{2}}a^{-\frac{1}{4}} + 10xa^{-\frac{1}{2}} - 3x^{\frac{1}{2}}a^{-\frac{3}{4}} + 9a^{-1} \text{ by } 3a^{-\frac{1}{4}}x^{\frac{1}{2}}.$$

Extract the square root of

$$35. x^{-2} + 4x^{-1} + 9 + 4x^{-\frac{3}{2}} + 6x^{-1} + 12x^{-\frac{1}{2}}.$$

$$36. 4x^3 + 4x^{\frac{5}{2}} - 3x^2 - 2x^{\frac{3}{2}} + x.$$

$$37. 23 + 4\sqrt{15}.$$

Solution: Make coefficient of radical 2.

$$23 + 4\sqrt{15} = 23 + 2\sqrt{60}, \text{ so that form is } (a + b) + 2\sqrt{ab}.$$

This is the square of $\sqrt{a} + \sqrt{b}$. By trial, find two numbers whose product is 60 and whose sum is 23. These are 20 and 3. Connect the square roots of these numbers by the sign of the radical term of the given binomial.

$$\text{The result is } \sqrt{20} + \sqrt{3} \text{ or } 2\sqrt{5} + \sqrt{3}.$$

$$38. 35 - 12\sqrt{6}.$$

$$40. 57 - 12\sqrt{15}.$$

$$39. 17 + 12\sqrt{2}.$$

$$41. 14 - 3\sqrt{3}.$$

Simplify and find the numerical value to the nearest hundredth.

$$42. \frac{3\sqrt{2} - \sqrt{3}}{5\sqrt{3} - 2\sqrt{6}}.$$

$$43. \frac{3 - 2\sqrt{5} + 3\sqrt{2}}{3 + 2\sqrt{5} - 3\sqrt{2}}.$$

$$44. \frac{5\sqrt{2} - 2\sqrt{3} + 2}{3\sqrt{2} + \sqrt{3} - 1}.$$

Hint: Multiply both numerator and denominator by $3 + 2\sqrt{5} + 3\sqrt{2}$, then simplify the resulting expression which has a binomial denominator.

9. Quadratic Equations are those which contain the second power of the unknown quantity but no higher power. Three methods of solution have been taught in your earlier courses. These are illustrated by the following examples.

1. *Method of Factoring.*

Illustrative example I:

Solve $24x^2 = 2x + 15$.

Solution: $24x^2 - 2x - 15 = 0$.

Factoring, $(6x - 5)(4x + 3) = 0$.

Then, $6x - 5 = 0$, or $4x + 3 = 0$.

Whence, $x = \frac{5}{6}$, or $-\frac{3}{4}$.

Why is it necessary that the right member be made zero before we factor?

What is the reason for setting $6x - 5$ and $4x + 3$ each equal to zero?

Can every quadratic equation be solved by this method? Why?

Can higher degree equations sometimes be solved by this method? If so, when?

2. *Method of Completing the Square.*

Illustrative example II:

Solve $6x^2 - x - 35 = 0$.

Transpose numerical term to right member and divide by 6.

Then,
$$x^2 - \frac{1}{6}x = \frac{35}{6}.$$

Add the square of half the coefficient of x to each member.

$$x^2 - \frac{1}{6}x + \frac{1}{144} = \frac{841}{144}.$$

Extract the square root of both members.

Whence
$$x - \frac{1}{12} = \pm \frac{29}{12}.$$
$$x = \frac{5}{2} \text{ or } -\frac{7}{3}.$$

Why is this process called "completing the square"?

Why must we place the $\pm$ sign before the square root of the right member?

Would we obtain additional roots if we placed the $\pm$ sign before the square root of both members?

Let the pupil state the complete rule for the above solution.

3. *Hindu Method.* Method number 2 is always usable but it sometimes gives us inconvenient fractions. In such cases the following method is preferable. Reduce the equation to the form

$$ax^2 + bx = c,$$

in which a and b are whole numbers, and *prime* to each other, but c may be integral or fractional.

Multiply each member of this equation by $4a$, and it becomes

$$4a^2x^2 + 4abx = 4ac.$$

Adding b^2 to each member, we have

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2.$$

Extracting the square root, we obtain

$$2ax + b = \sqrt{4ac + b^2}.$$

Transposing b , and dividing by $2a$,

$$x = \frac{-b \pm \sqrt{4ac + b^2}}{2a}.$$

If b is an even number, $\frac{b}{2}$ will be an integral number; and it would have been sufficient to multiply each member by a , and add $\frac{b^2}{4}$ to each member. Hence we have the

Rule. — (1) *Reduce the equation to the form $ax^2 + bx = c$, where a and b are prime to each other.*

(2) *If b is an odd number, multiply the equation by four times the coefficient of x^2 , and add to each member the square of the coefficient of x .*

(3) *If b is an even number, multiply the equation by the coefficient of x^2 , and add to each member the square of half the coefficient of x .*

4. Method of the Quadratic Formula.

The quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is obtained by solving the equation $ax^2 + bx + c = 0$ by the method of completing the square.

Illustrative example III:

Solve by formula $4x^2 + 8x - 5 = 0$.

Solution: $a = 4, b = 8, c = -5$.

Hence,

$$x = \frac{-8 \pm \sqrt{64 + 80}}{8} = \frac{-8 \pm 12}{8} = \frac{1}{2} \text{ or } -\frac{5}{2}.$$

Why may $ax^2 + bx + c = 0$ be regarded as representing any quadratic equation?

Exercises

Solve and check as directed:

1. $ax^2 + bx + c = 0$ by completing the square.

2. $\frac{x^2 - 5}{x^2 + x - 6} + \frac{1}{x - 2} = \frac{1}{x + 3}.$
3. $(3y + 1)^2 + (1 - 2y)(4y + 1) - 11 = 0.$
4. $\frac{4}{x - 4} = \frac{x^2 - 18}{x^2 - 16} + \frac{5}{x + 4}.$
5. $\frac{x^2}{2} - \frac{x}{3} + 7\frac{1}{2} = 8\frac{1}{8}.$
6. $\frac{1}{2x - 1} - \frac{1}{8x - 1} = \frac{3}{8x^2 - 2x - 1}.$
7. $\frac{x}{a} - \frac{b}{a} = \frac{a}{b} - \frac{a}{x}.$
8. $\frac{x}{m + x} - \frac{n}{n + m} - \frac{n}{n + p} = -\frac{x}{p + x}.$
9. $\frac{1}{x} + \frac{1}{c} + \frac{1}{d} = \frac{1}{x + c + d}.$
10. $\frac{1}{x - a} + \frac{1}{a} - \frac{1}{b} = \frac{1}{x - b}.$
11. $\frac{2x - 1}{x + 1} - 1 = \frac{3x - 4}{x - 1} - \frac{x - \frac{7}{2}}{\frac{1}{4} - \frac{x^2}{4}}.$
12. $\frac{y + 5}{y^2} = \frac{m + 5}{m^2}.$
13. $(a + b)^2x^2 - (a^2 - b^2)x = ab.$
14. $x^2 + c^2 + x = 6 + c + 2cx.$
15. $\frac{m - n + x}{m + n + x} = 2 - \frac{m + n}{x + n}.$
16. $\frac{b}{2a}x^2 + x - 1 = \frac{b}{2a}.$
17. $(x^2 - 1)(a^2 - b^2) + 4abx = 0.$
18. $.6x^2 - 2.24x + 3.16 = 0.$

$$19. (a^2 - b^2)x^2 - 2(a^2 + b^2)x + a^2 - b^2 = 0.$$

$$20. a^2x^2 - 2a^3x + a^4 - 1 = 0.$$

$$21. a(b - c)x^2 + b(c - a)x + c(a - b) = 0.$$

$$22. (x^2 + 2x)(x^2 - 2x - 3)(x^2 - x + 1) = 0.$$

$$23. x^2 - 6acx + a^2(9c^2 - 4b^2) = 0.$$

$$24. (4a^2 - 9b^2)(x^2 + 1) = 2x(4a^2 + 9b^2).$$

$$25. 2.5x^2 - 3.4x = -0.48.$$

$$26. (x - 3)(x + 2) - (x - 3)(x^2 - 5x + 4) = 0.$$

$$27. \frac{15 - 4x - x^2}{1 - x^2} = \frac{2x - 1}{x + 1} - \frac{3x - 4}{x - 1}.$$

$$28. \frac{x + 7}{2x^2 - 7x + 3} + \frac{x}{x^2 - 2x - 3} - \frac{x + 3}{1 - x - 2x^2} = 0.$$

$$29. .03y^2 - 2.23y = -1.1075.$$

$$30. \frac{x^4 + 2x^3 + 8}{x^2 + x - 6} = x^2 + x + 8.$$

$$31. \frac{5}{5x - 1} - \frac{26}{(x + 5)(5x - 1)} = \frac{1}{3x - 5}.$$

$$32. \frac{2x^2 - 4x - 3}{2x^2 - 2x + 3} = 1 + \frac{2 - x}{x^2 - 3x - 2}.$$

$$33. \frac{3y^2 - 4}{y^2 + 5} - \frac{4y^2 + 3}{y^2 - 5} + \frac{2y^4 + 12}{y^4 - 25} = 1.$$

$$34. \frac{3}{y - 2} - \frac{2}{3 - y} = \frac{4}{y - 1} - \frac{1}{4 - y}.$$

$$35. 49x^4 - 70x^2 + 12 = 0, \text{ by formula to three decimal places.}$$

10. **The Quadratic Form.** — An equation such as

$$2(x^2 - 1)^2 - 5(x^2 - 1) - 3 = 0$$

is said to be in the *quadratic form*, though it is not a quadratic equation. However, the quantity $(x^2 - 1)$ enters into the

equation in the same way that x enters into the true quadratic. We may, therefore, solve it for $(x^2 - 1)$ by the quadratic formula.

$$\text{Thus, } (x^2 - 1) = \frac{5 \pm \sqrt{25 + 24}}{4} = \frac{5 \pm 7}{4} = 3 \text{ or } -\frac{1}{2}.$$

$$\begin{array}{ll} \text{Hence, } x^2 - 1 = 3. & x^2 - 1 = -\frac{1}{2}. \\ x^2 = 4. & x^2 = \frac{1}{2}. \\ x = \pm 2. & x = \pm \frac{1}{2}\sqrt{2}. \end{array}$$

Illustrative example II:

$$\text{Solve } 8x^2 - 4x^{-2} - 31 = 0.$$

$$\text{Solution: } 8x^2 - \frac{4}{x^2} - 31 = 0.$$

$$8x^4 - 31x^2 - 4 = 0.$$

$$x^2 = \frac{31 \pm \sqrt{961 + 128}}{16}$$

$$= \frac{31 \pm 33}{16} = 4 \text{ or } -\frac{1}{8}.$$

$$x = \pm 2 \text{ or } \pm \frac{1}{4}\sqrt{-2}.$$

11. Irrational Equations contain the unknown quantity under a radical sign or with a fractional exponent. Sometimes they are in the quadratic form and may be solved accordingly. In other cases the equation must be made rational by raising both members to the same power.

Illustrative example I:

$$\text{Solve } \sqrt{5x - 4} - \sqrt{2x + 1} = \sqrt{x - 3}.$$

Solution: By squaring both members of the equation we obtain,

$$5x - 4 - 2\sqrt{10x^2 - 3x - 4} + 2x + 1 = x - 3.$$

$$-2\sqrt{10x^2 - 3x - 4} = -6x.$$

$$\sqrt{10x^2 - 3x - 4} = 3x.$$

$$\begin{aligned}
 \text{Again squaring,} \quad & 10x^2 - 3x - 4 = 9x^2. \\
 & x^2 - 3x - 4 = 0. \\
 & (x - 4)(x + 1) = 0. \\
 & x = 4, -1.
 \end{aligned}$$

Both results check.

Illustrative example II:

$$\text{Solve} \quad x = 6 + \sqrt{3x}.$$

$$\text{Solution:} \quad x - 6 = \sqrt{3x}.$$

$$x^2 - 12x + 36 = 3x.$$

$$x^2 - 15x + 36 = 0.$$

$$(x - 12)(x - 3) = 0.$$

$$x = 12, 3.$$

Here 3 does not satisfy the equation, and 12 is the only root. Values that do not check in the equation, such as 3, are introduced in the process of squaring, and are called *extraneous*. They are roots of the squared equation, but not of the original equation. You should always check the solution of irrational equations, and reject extraneous results.

Exercises

Solve and check as directed:

$$1. \quad x^4 - 3x^2 = 18. \quad 2. \quad x^6 - 7x^3 - 8 = 0.$$

$$3. \quad (x - 1)^2 - 4(x - 1) = 5.$$

$$4. \quad (mx + n)^2 + 3(mx + n) = -2.$$

$$5. \quad \left(\frac{1}{x} - 1\right)^2 - 3\left(\frac{1}{x} - 1\right) + 2 = 0.$$

$$6. \quad (x - 1) + 3\sqrt{x - 1} = 10.$$

$$7. \quad x - 3x^{\frac{1}{2}} = 4. \quad 8. \quad 3x^{\frac{2}{3}} - 4x^{\frac{1}{3}} - 4 = 0.$$

$$9. \quad 3x^{-1} - x^{-\frac{1}{2}} = 10.$$

$$10. \left(\frac{x}{2} + \frac{1}{x}\right)^2 - 3\left(\frac{x}{2} + \frac{1}{x}\right) + \frac{9}{4} = 0.$$

$$11. 7x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} + 4 = 0.$$

$$12. x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12.$$

Hint: Add 3 to each member of the equation, thus,

$$(x^2 - 5x + 3) + 2\sqrt{x^2 - 5x + 3} = 15.$$

$$13. 3x^2 - 16x + 3 + 3\sqrt{3x^2 - 16x + 21} = 10.$$

$$14. \sqrt{27x + 1} = 2 - 3\sqrt{3x}.$$

$$15. \sqrt{2x - 3a} + \sqrt{3x - 2a} = 3\sqrt{a}.$$

$$16. \sqrt{2x + 7} - \sqrt{7x + 1} = -\sqrt{3x - 18}.$$

$$17. x^2 + 8x - \sqrt{4x^2 + 32x + 12} = 21.$$

$$18. (x^2 + 2)^{\frac{5}{2}} + \frac{3}{(x^2 + 2)^{\frac{1}{2}}} = 4(x^2 + 2).$$

$$19. 3x^{-\frac{3}{2}} + 20x^{-\frac{3}{4}} = 32.$$

$$20. \sqrt{7 + 4x} + 3\sqrt{2x^2 + 5x + 7} = 3.$$

$$21. 2\sqrt{2x + 2} - \sqrt{2x + 1} = \frac{4(3x + 1)}{\sqrt{8x + 8}}.$$

$$22. 4x - 3x^2 + 2\sqrt{3x^2 - 7x + 3} = -3x.$$

$$23. x^2 - 4ax - 18a^2 + 6a\sqrt{x^2 - 4ax + 9a^2} = 0.$$

$$24. 6\sqrt{x^2 - 2x + 6} = 21 + 2x - x^2.$$

$$25. \sqrt{3x + 1} + \frac{35}{\sqrt{3x + 1}} = 3\sqrt{x}.$$

$$26. (x^2 - 1)^2 - 11(x^2 - 1) + 24 = 0.$$

$$27. x^2 - 2\sqrt{x^2 + 16} = -1.$$

12. Systems of Equations. — Two or more equations in the same number of unknown quantities form a system when they are considered together for the purpose of obtaining a common set of roots.

In order that it may be possible to solve such a system it is necessary that the equations forming it shall be *consistent* and *independent*. In that case the system is called *simultaneous*. Such a system has a single set of roots, or at most, sets of roots whose number does not exceed the product of the degrees of the equations.

A system in which one equation can be formed from another by multiplying it by a constant is an *indeterminate* or *dependent* system. Such a system is satisfied by an indefinite number of values of the unknowns. An example is

$$3x + 2y = 8.$$

$$\frac{3}{2}x + 3y = 12.$$

In this case any set of values which will satisfy one equation will also satisfy the other. $x = 0, y = 4$; $x = 2, y = 1$; and $x = -2, y = 7$, are three such sets. Let the pupil determine several more sets.

A system in which the statement made by one equation contradicts the statement made by the other is an *inconsistent* system. Such a system has no roots. An example is

$$3x - y = 10.$$

$$6x - 2y = 5.$$

Here no value of x and y can be found which will satisfy both of these equations, for if $3x - y = 10$, then $6x - 2y = 20$, and the statement that $6x - 2y = 5$ is, therefore, inconsistent with the first statement.

13. Elimination. — The solution of systems of equations is effected by eliminating all of the unknowns but one, and then solving the resulting equation for the value of that unknown. The values of the remaining unknown can then usually be found by substitution.

The usual methods of elimination are *addition or subtraction*, and *substitution*, and these will be illustrated in the examples which follow.

Illustrative example I:

Solve $3x - 2y = 7. \quad (1)$

$5x + 7y = 20. \quad (2)$

Solution:

$21x - 14y = 49.$ Multiply equation (1) by 7.

$10x + 14y = 40.$ Multiply equation (2) by 2.

$\underline{31x = 89.}$ By adding.

$x = \frac{89}{31}.$

Then $\frac{287}{31} - 2y = 7.$

$267 - 62y = 217.$

$-62y = -50.$

$y = \frac{25}{31}.$

Illustrative example II:

Solve $3x + 8y = -5.$

$x^2 - 2xy = 12.$

Solution: Solve the first equation for the value of x .

Then, $x = \frac{-5 - 8y}{3}.$

Substitute this value for x in the second equation and we have

$$\left(\frac{-5 - 8y}{3}\right)^2 - 2y\left(\frac{-5 - 8y}{3}\right) = 12.$$

$$\frac{25 + 80y + 64y^2}{9} - \frac{-10y - 16y^2}{3} = 12.$$

$$25 + 80y + 64y^2 + 30y + 48y^2 = 108.$$

$$112y^2 + 110y - 83 = 0.$$

$$(2y - 1)(56y + 83) = 0.$$

$$y = \frac{1}{2} \text{ or } -\frac{83}{56}.$$

$$x = -3 \text{ or } \frac{19}{7}.$$

That is, $x = -3$ and $y = \frac{1}{2}$, or $x = \frac{16}{7}$ and $y = -\frac{83}{56}$.

In the examples which follow some systems are linear, while others are quadratic. For a discussion of the methods applicable to the various types of quadratic systems, see the authors' Second Course in Algebra, Chapter XI, or Intermediate Algebra, Chapter X.

Exercises

Solve each of the following systems of equations, correctly group the sets of roots, and check as directed:

1. $3x - 2y = 4.$

$6x + 5y = 39\frac{1}{2}.$

2. $\frac{3x}{5} + \frac{1-3y}{7} = \frac{11}{5}.$

$\frac{3y}{11} = 9 - \frac{12x}{11}.$

3. $.3x + 4y = 18.2.$

$.09x - .7y = 3.96.$

6. $\frac{1 - .1y + 2.5x}{x + y - 10} = \frac{5}{2}.$

$\frac{.5}{.4x - 1.1} = \frac{4}{1.1y - 7}.$

7. $a^2x - b^2y = a - b.$

$2ab^2x - a^2by = 2a^2 - b^2.$

8. $\frac{x+y}{ab} = \frac{1}{b} + \frac{1}{a}.$

$a(x-y) = \frac{a^2 - b^2}{b}.$

9. $\frac{s}{x} + \frac{t}{y} = m.$

$\frac{2s}{x} - \frac{5t}{y} = 3n.$

4. $\frac{2}{x} - \frac{10}{y} = -19.$

$\frac{7}{x} + \frac{3}{y} = 28\frac{1}{2}.$

5. $\frac{1}{y} + \frac{1}{x} = -\frac{1}{6}.$

$\frac{3}{y} - \frac{2}{x} = -\frac{4}{3}.$

$$\begin{aligned} 10. \quad & 3x + 5y + 2z = 8. \\ & 7x - y - 6z = 4. \\ & -6x - 3y + 8z = 24. \end{aligned}$$

$$\begin{aligned} 11. \quad & 2x + 9y = a + 6b. \\ & 3y - 8z = 2b - 5c. \\ & 10x + z = \frac{40a + 5c}{8}. \end{aligned}$$

$$\begin{aligned} 12. \quad & \frac{l}{l+x} + \frac{m}{m-y} = \frac{l}{m}. \\ & \frac{m}{l+x} + \frac{l}{m-y} = \frac{m}{l}. \end{aligned}$$

$$\begin{aligned} 13. \quad & xy - y^2 = 2. \\ & x^2 + 2y^2 = 17. \end{aligned}$$

$$\begin{aligned} 14. \quad & 3xy + 3x = 12. \\ & 3x - 2y = 0. \end{aligned}$$

$$\begin{aligned} 15. \quad & x^2 + 4xy = 57. \\ & x + y = 7. \end{aligned}$$

$$\begin{aligned} 16. \quad & x^2y^2 + xy - 12 = 0. \\ & x + y = 4. \end{aligned}$$

$$\begin{aligned} 17. \quad & 4x^2 + 3xy - 2y^2 = 18. \\ & 3x^2 + 2xy - y^2 = 27. \end{aligned}$$

$$\begin{aligned} 18. \quad & x^2 + xy + x = 14. \\ & y^2 + xy + y = 28. \end{aligned}$$

$$\begin{aligned} 19. \quad & x^2 + y^2 = 4. \\ & x = (1 + \sqrt{2})y - 2. \end{aligned}$$

$$20. \quad \frac{1}{x} + \frac{1}{y} = 5.$$

$$\frac{1}{x^2} + \frac{1}{y^2} = 13.$$

$$\begin{aligned} 28. \quad & xy^2(xy^2 - 6) + 9 = 0. \\ & xy = 2 + y. \end{aligned}$$

$$21. \quad \frac{1}{x^3} + \frac{1}{y^3} = \frac{243}{8}.$$

$$\frac{1}{x} + \frac{1}{y} = \frac{9}{2}.$$

$$\begin{aligned} 22. \quad & x^2y^2 + 28xy = 480. \\ & y = 11 - 2x. \end{aligned}$$

$$\begin{aligned} 23. \quad & x^{-\frac{1}{2}} + 2y^{-\frac{1}{2}} = \frac{7}{6}. \\ & 2x^{-\frac{1}{2}} - y^{-\frac{1}{2}} = \frac{2}{3}. \end{aligned}$$

$$\begin{aligned} 24. \quad & x^3 - y^3 = 127. \\ & x^2y - xy^2 = 42. \end{aligned}$$

$$\begin{aligned} 25. \quad & x^2 + xy + y^2 = 84. \\ & x + \sqrt{xy} + y = 14. \end{aligned}$$

$$\begin{aligned} 26. \quad & x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5 \\ & x^{-\frac{1}{2}} + y^{-\frac{1}{2}} = \frac{5}{8}. \end{aligned}$$

$$\begin{aligned} 27. \quad & x^4 + y^4 = 97. \\ & xy = -6. \end{aligned}$$

$$\begin{array}{ll}
 29. & 0.1x + 0.3y = 1.9. \\
 & 0.2x + 0.4z = 3.2. \\
 & 0.5y + 0.1z = 3.1
 \end{array}
 \qquad
 \begin{array}{l}
 30. \quad x^2 + 4y^2 = 4a^2 + b^2. \\
 \quad \quad 2x + y = 2b - a.
 \end{array}$$

$$\begin{array}{l}
 31. \quad 4x^2 + 9y^2 + 21y = 30. \\
 \quad \quad 2x - 3y = 12.
 \end{array}$$

$$\begin{array}{l}
 32. \quad (a - b)x + (a + b)y = 2(a^2 - b^2). \\
 \quad \quad ax - by = a^2 + b^2.
 \end{array}$$

$$\begin{array}{l}
 33. \quad \quad \quad x^2 + 2y^2 = 6. \\
 \quad 2(x - 3)^2 + (2y + 1)^2 = 11.
 \end{array}$$

$$\begin{array}{ll}
 34. \quad x^2 + xy = ay. & 35. \quad \quad \quad x - y = a + b. \\
 (a - x)y = 2ax. & \quad \quad \quad \frac{x - a}{a} + \frac{y - a}{b} = 0
 \end{array}$$

$$\begin{array}{l}
 36. \quad (x + y) : (y - x) = a : 1. \\
 \quad \quad (xy - 3) : (x - 1) = (a + 2) : 1.
 \end{array}$$

$$\begin{array}{l}
 37. \quad \quad \quad y : x = (a^3 + b^3) : (a^3 - b^3). \\
 \quad \quad (y - a) : (x - a) = (a + b) : (a - b).
 \end{array}$$

$$\begin{array}{ll}
 38. \quad \frac{x}{a} + \frac{y}{b} - 1 = 0. & 39. \quad 16x - \frac{.5w}{y} = 4. \\
 \quad \quad \frac{x}{b} - \frac{y}{a} - 1 = 0. & \quad \quad 9x - \frac{.3}{y} = 5.
 \end{array}$$

40. Show that the following equations are dependent and find two pairs of values of x and y that satisfy them:

$$\begin{array}{l}
 19.6x + 26.25y = 22.4. \\
 2.8x + 3.75y = 3.2.
 \end{array}$$

14. Problems. — Practice in solving applied problems is constantly necessary for growth in mathematical power. In the examples which follow you may use any method which seems best adapted to the problem.

Exercises

1. A rectangle is surrounded by a border whose width is 5 inches. The area of the rectangle is 168 square inches, and that of the border is 360 square inches. Find the length and breadth of the rectangle.

2. After street improvement it is found that a certain corner rectangular lot has lost $\frac{1}{10}$ of its length and $\frac{1}{15}$ of its width. Its perimeter has been decreased by 28 feet and the new area is 3024 square feet. Find the reduced dimensions of the lot.

3. Both the numerator and the denominator of a fraction are increased by their squares. The resulting fraction reduces to $\frac{1}{2}$. The sum of the numerator and the denominator is 5. What is the fraction?

4. \$4400 is invested in two parts and at different rates of interest so as to give the same income. If the first part were invested at the second rate, the income from this part would be \$100; but if the second part were invested at the first rate, its income would be \$144. What are the rates of interest?

5. A man derives an income of \$165 per year from some money invested at 3% and some at $3\frac{1}{2}\%$. If the amounts of the respective investments were interchanged, he would receive \$160. How much has he in each investment?

6. A man bought a certain number of oranges for \$2. If he had paid 5 cents more per dozen, he would have received two dozen less for the same money. How much did he pay for a dozen, and how many oranges did he buy?

7. A man bought a certain number of grapefruit for \$1.04. After throwing away 4 bad ones, he sold the others for 6 cents apiece more than he paid for them and made a profit of 22 cents. How many did he buy, and what did he pay for each?

8. There is a number consisting of two digits. The left-hand digit is equal to three times the right-hand digit; and if 12 be subtracted from the number itself, the remainder will be equal to the square of the left-hand digit. What is the number?

9. There are three numbers, the difference of whose differences is 8. Their sum is 41 and the sum of their squares is 669. What are the numbers?

10. There are three numerical quantities having the following relations to each other: the sum of the squares of the first and second, added to the first and second, is 32; the sum of the squares of the first and third, added to the first and third, is 42; and the sum of the squares of the second and third, added to the second and third, is 50. Find the quantities.

11. Find two quantities such that their sum, their product, and the sum of their squares shall all be equal to each other.

12. A and B run a mile race on two occasions. In the first race A gives B a start of 10 yards and beats him by 10 seconds. In the second race A gives B a start of 11 seconds and beats him by 5 yards. Find the rate at which each runs.

13. A starts from Rochester at 8 A.M. on a bicycle for Syracuse and return at the same time that B starts from Syracuse for Rochester and return. The distance between the cities is 80 miles. They meet the first time at noon and the second time at a point 16 miles from Syracuse. Find the rate at which each rode.

14. A cask is half full of wine and an equal-sized cask is half full of water. A gallon of wine is removed from the wine cask and poured into the water cask, after which a

gallon of the mixture is removed from the latter cask and poured back into the wine cask. As a result of these two operations, has more wine been removed from the wine cask or water from the water cask, and how much?

15. Two dealers bought together 50 head of cattle for \$1600, A paying for each as many dollars as the number of head B bought. If B had bought his at the price A paid for his, and A paid B's price, they would together have paid \$1900. How many cattle did each buy and at what price?

16. The front wheel of a wagon makes 110 more revolutions than the rear wheel in going a mile. If the circumference of the front wheel were increased 25% and that of the rear wheel decreased 25%, the rear wheel would revolve 88 more times than the front wheel in going the same distance. Find the circumferences of the front and rear wheels.

17. A soldier was sent with a message from the rear line of an army to an officer in the first line. He delivered the message and immediately returned to the rear line, which at the time he reached it, occupied the position which was occupied by the front line when he set out to deliver the message one hour earlier. If the army was marching in column four miles deep, how far did the messenger ride?

18. Pure ground mustard contains 21.5% of oil. A sample of mustard which has been adulterated with flour is found by analysis to contain 18.8% of oil. If the flour with which the mustard was adulterated contains 2.2% of oil of its own, what was the percentage of flour in the sample?

19. There are two pumps drawing water from a tank. When the first works three hours and the second five hours, 1350 cubic feet of water are withdrawn. When the first

works four hours and the second three hours, 1250 cubic feet of water are withdrawn. How many cubic feet of water can each pump discharge in one hour?

20. A steamer makes a trip of 70 miles up a river and down again in 24 hours, allowing 5 hours for taking on a cargo. It is observed that it requires the same time to go $2\frac{1}{2}$ miles up the river as 7 miles down the river. Find the number of hours required for the up trip and for the down trip respectively.

CHAPTER II

RATIO, PROPORTION, VARIATION, AND PROGRESSIONS

15. A Ratio is the indicated quotient of two like magnitudes; as, $\frac{2}{3}$, $\frac{a}{b}$, $\frac{2 \text{ quarts}}{9 \text{ quarts}}$, etc.

16. A Proportion is a statement that two ratios are equal.

Example, $\frac{a}{b} = \frac{x}{y}$. This may also be written $a : b = x : y$.

17. Proportionals.—The quantities which make up a proportion are called *proportionals*.

In the proportion $a : b = c : x$, x is the *fourth proportional* to a , b , and c .

In the proportion $a : b = b : x$, x is the *third proportional* to a and b .

In the proportion $a : x = x : b$, x is the *mean proportional* between a and b .

18. Means are the second and third terms of a proportion; **extremes** are the first and fourth terms.

19. Antecedents are the first and third terms of a proportion; **consequents** are the second and fourth terms.

20. Theorems.—The following are well-known theorems concerning ratio and proportion with which the pupil is no doubt familiar. He should be required to give an algebraic proof of each of them.

1. *The value of a ratio is unaltered if both antecedent and consequent are either multiplied or divided by the same number.*

2. In a proportion, both antecedents or both consequents may be either multiplied or divided by the same number, without destroying the truth of the proportion.

3. If the antecedents of a proportion are equal, the consequents also are equal, and conversely.

4. In any proportion the product of the means is equal to the product of the extremes.

5. The mean proportional between two quantities is equal to the square root of their product.

6. If the product of two quantities equals the product of two others, a proportion may be written in any way which will make one pair the means and the other pair the extremes.

7. If three terms of one proportion are respectively equal to the corresponding three terms of another, the fourth terms also are equal.

8. Like powers, or like roots, of the terms of a proportion are in proportion.

21. Proportion Transformations. — Let the pupil prove :

1. If four quantities are in proportion, they are in proportion by alteration, that is, the first term is to the third as the second term is to the fourth. Ex., if $a : b = c : d$, then $a : c = b : d$.

2. If four quantities are in proportion, they are in proportion by inversion, that is, the second is to the first as the fourth is to the third. Ex., if $a : b = c : d$, then $b : a = d : c$.

3. If four quantities are in proportion, they are in proportion by addition, that is, the sum of the first two terms is to the first term (or second) as the sum of the last two terms is to the third (or fourth). Ex., if $a : b = c : d$, then $(a + b) : b = (c + d) : d$.

4. If four quantities are in proportion, they are in proportion by subtraction, that is, the difference of the first two terms is to the first (or second) as the difference of the last two is to the third (or fourth). Ex., if $a : b = c : d$, then $(a - b) : b = (c - d) : d$.

5. If four quantities are in proportion, they are in proportion by addition and subtraction, that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference. Ex., if $a : b = c : d$, then $(a + b) : (a - b) = (c + d) : (c - d)$.

Hint: In proving the first two of the above transformations you may find it helpful to use theorems 4 and 6. In proving the third transformation add 1 to each member of the equation $\frac{a}{b} = \frac{c}{d}$. Similarly, subtract 1 from each member of the same equation in proving the fourth transformation. To prove the fifth transformation, divide the proportion resulting in (3) by that resulting in (4).

22. Theorem. — (9) *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents, as any antecedent is to its consequent.*

Given $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}$, to prove $\frac{a + c + e + g}{b + d + f + h} = \frac{a}{b} = \frac{c}{d}$, etc.

Proof: Since $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, etc., are all equal, denote the value of each by r .

$$\text{Then,} \quad \frac{a}{b} = r, \frac{c}{d} = r, \frac{e}{f} = r, \frac{g}{h} = r.$$

$$\therefore \quad a = br.$$

$$c = dr.$$

$$e = fr.$$

$$g = hr.$$

$$\text{Adding,} \quad a + c + e + g = br + dr + fr + hr \\ = r(b + d + f + h).$$

$$\text{Hence, } r = \frac{a + c + e + g}{b + d + f + h}. \quad \text{But, } r = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}, \text{ etc.}$$

$$\text{Hence, } \frac{a + c + e + g}{b + d + f + h} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}, \text{ etc.}$$

23. Direct and Inverse Proportion. — Two variable quantities are said to be *directly proportional* if the ratios of any two values of the one variable is the same as the ratio of the corresponding values of the other variable. For example, if a train is moving at a uniform rate, the distance it has gone is directly proportional to the time it has been going. Let the rate be 40 miles per hour. Then in 2 hours the train will go 80 miles; in 5 hours 200 miles.

Hence, $\frac{2}{5} = \frac{80}{200}$.

Two variable quantities are said to be *inversely proportional* if the ratio of any two values of the one variable is equal to the reciprocal of the ratio of the corresponding values of the other variable. For example, the length of a rectangle having a fixed area is inversely proportional to its width. Let the area be 60 square inches.

Let $l = 12$, then $w = 5$.

Let $l = 10$, then $w = 6$.

And $\frac{12}{10} = \frac{6}{5}$ (the reciprocal of $\frac{5}{6}$).

Illustrative example I:

If $a : b = c : d$, prove that

$$a^2 : b^2 = (a^2 + c^2) : (b^2 + d^2).$$

$$\text{Proof:} \quad \frac{a}{b} = \frac{c}{d} \quad \text{Given.}$$

$$\frac{a^2}{b^2} = \frac{c^2}{d^2} \quad \text{Theorem 8.}$$

$$\frac{a^2}{c^2} = \frac{b^2}{d^2} \quad \text{Transformation I.}$$

$$\frac{a^2 + c^2}{a^2} = \frac{b^2 + d^2}{b^2} \quad \text{Transformation III.}$$

$$\frac{a^2 + c^2}{b^2 + d^2} = \frac{a^2}{b^2} \quad \text{Transformation I.}$$

$$\text{Or} \quad a^2 : b^2 = (a^2 + c^2) : (b^2 + d^2).$$

Illustrative example II :

Solve $(x^2 + x + 1) : (x^2 - x - 1) = (x + 2) : (x - 2)$.

By transformation V,

$$\frac{2x^2}{2x + 2} = \frac{2x}{4}.$$

$$\frac{x^2}{x + 1} = \frac{x}{2}.$$

Then,

$$2x^2 = x^2 + x,$$

and

$$x = 0, \text{ or } 1.$$

Illustrative example III :

The weight of a body above the earth's surface is inversely proportional to the square of its distance from the center of the earth. If a boy at the surface of the earth weighs 120 pounds, what will he weigh 6000 miles above the earth's surface? Take r (radius of earth) as 4000 miles.
 $4000 + 6000 = 10,000$.

$$\text{Solution:} \quad 10,000^2 : 4000^2 = 120 : x.$$

$$100 : 16 = 120 : x.$$

$$100x = 1920.$$

$$x = 19.2 \text{ pounds.}$$

Exercises

1. Show that a ratio is always a pure (abstract) number.
2. If $5x - 2y = 3x + 5y$, find the ratio of x to y .
3. Find a fourth proportional to $5\frac{1}{2}$, 8, and $13\frac{3}{4}$.
4. Find a third proportional to 3 and 15; to x and y .
5. Find a mean proportional between $81x^2$ and $\frac{16y^2}{9x^4}$;
 between $(a - x)(a^2 - x^2)$ and $(a + x)^3$.

Solve the equations :

• 6. $(5x - 1) : (2x + 1) = (3x + 1) : (x + 2)$.

$$7. (x^2 - 4) : (x^2 + 2) = (x + 2) : (2x + 1).$$

$$8. (x^2 + 5x - 2) : (x^2 - 5x + 2) = (x + 3) : (x - 3).$$

$$9. \sqrt{3x + 1} : \sqrt{x - 1} = \sqrt{8x - 4} : \sqrt{x + 4}.$$

If $a : b = c : d$, prove that

$$10. c : d = \sqrt{a^2 + 5c^2} : \sqrt{b^2 + 5d^2}.$$

$$11. ac : bd = (a^2 - c^2) : (b^2 - d^2).$$

$$12. (ac - b) : (c^2 - d) = a : c.$$

$$13. \text{ If } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{g}{h}, \text{ prove that}$$

$$\frac{a + 3c - 5e + 10g}{b + 3d - 5f + 10h} = \frac{c}{d}.$$

14. Find two numbers in the ratio of 3 to 5 such that if 6 and 60 are added to them respectively, they will be in the ratio of 3 to 10.

15. Divide 60 into two parts such that the greater increased by 4 is to the less diminished by 4 as 3 is to 2.

16. If the grain I have on hand will last 30 cattle 10 weeks, how long will it last 20 cattle? Is this a direct or inverse proportion?

17. If it takes 8 men 20 days to do a piece of work, how long will it take 6 men to do it?

18. The distance a body falls from rest is directly proportional to the square of the time it has been falling. If a body falls 64 feet in 2 seconds, how far will it fall in 5 seconds?

19. Separate a line 56 inches long into two parts which have the ratio 3 to 4.

20. The corresponding sides of two similar triangles are 6 and 10. If the area of the first is 60 square inches, what is the area of the second?

21. The sales of a store were \$50,000. The net profits on these sales were \$3250. What would be the net profits on sales of \$88,000 at the same rate?

22. The areas of two similar triangles are to each other as the squares of any two corresponding sides. Given a triangle whose area is 324, and another similar triangle with area of 78. The side of the larger triangle is 15. What is the corresponding side of the smaller triangle?

24. **Variation** is a method of treating related numbers which is closely associated with the method of proportion. Many examples may be solved by either method.

When two variable quantities are directly proportional, either is said to *vary directly* as the other. For example, when a body is moving at a fixed rate, the distance varies as the time. We write this $d \propto t$. ($\propto$, the sign of variation, is read "varies as.") In such a case, if d and d' are two values of d , and t and t' are the corresponding values of t , we may write the direct proportion

$$\frac{d}{d'} = \frac{t}{t'}. \quad \text{Hence, } \frac{d}{t} = \frac{d'}{t'}.$$

The ratio of $\frac{d}{t}$ is then constant. We call this the *constant of variation*. If we denote it by k , we have $\frac{d}{t} = k$ or $d = kt$.

Hence *every variation may be written as an equation* by the introduction of an arbitrary constant.

25. **Inverse Variation.** — One quantity is said to *vary inversely* as another, when the quantities are inversely proportional. In this case the product of the quantities is constant and one varies directly as the reciprocal of the other. For example, the volume of a gas under constant tem-

perature varies inversely as its pressure. That is, $v \propto \frac{1}{p}$. This may also be written $v = \frac{k}{p}$, $vp = k$, or by the proportion $\frac{v}{v'} = \frac{p'}{p}$.

26. Joint Variation. — One quantity is said to vary jointly as two or more other quantities when the first quantity varies directly as the product of these quantities. For example, the area of a triangle varies jointly as its base and altitude. That is, $A \propto bh$. This may obviously be written $A = kbh$. From your knowledge of geometry you will recognize the fact that in this case $k = \frac{1}{2}$.

Illustrative examples :

1. If $x \propto y^2$, and $x = 4$ when $y = 8$, find x when $y = 2$.

Solution: If $x \propto y^2$, then $x = ky^2$.

Then, since $x = 4$ when $y = 8$, $4 = k(8)^2$, and $k = \frac{1}{16}$.

Now if $y = 2$, we have $x = \frac{1}{16}(2)^2 = \frac{1}{4}$.

2. If x varies inversely as y , and $x = 3$ when $y = 8$, find x when $y = 6$.

Solution: $x \propto \frac{1}{y}$. Hence, $x = \frac{k}{y}$ and $xy = k$.

Hence, $3 \cdot 8 = k$ and $k = 24$.

Then, $y = 6$, $6x = 24$, and $x = 4$.

3. The pressure of a gas varies jointly as its density and its absolute temperature. When the density is 1 and the temperature is 300° the pressure is 15. What is the pressure when the density is 3 and the temperature is 320° ?

Solution: $P \propto TD$, that is, $P = kTD$.

Then, $15 = k \cdot 300 \cdot 1$. Hence, $k = \frac{1}{20}$.

When $D = 3$ and $T = 320$, we have

$$P = \frac{1}{20} \cdot 320 \cdot 3 = 48.$$

Exercises

1. Express each of the following relations, first by use of the variation sign; second by a proportion; third as an equation.

(a) The area of a parallelogram varies jointly as its base and altitude.

(b) The volume of a pyramid varies jointly as its base and altitude.

(c) The area of a circle varies as the square of its radius.

(d) The circumference of a circle varies as its radius.

(e) The volume of a sphere varies as the cube of its radius.

(f) The velocity of a falling body varies as the time it has been falling.

(g) The intensity of light on a screen is inversely proportional to the square of the distance of its source.

(h) The attraction between two bodies is inversely proportional to the square of the distance between them.

2. In the first six of the above variations state from your knowledge of geometry or physics the value of the constant of variation.

3. If $x \propto y$, and $x = 75$ when $y = 5$, what is the value of x when $y = 3$?

4. If $x \propto$ inversely as y , and $x = 13$ when $y = 9$, what is the value of x when $y = 3$?

5. If x varies jointly as y and z , and $x = 24$ when $y = 2$ and $z = 3$, find x when $y = 3$ and $z = 5$.

6. If $x \propto$ inversely as y , and $x = \frac{3}{4}$ when $y = \frac{2}{3}$, what is the value of x when $y = \frac{5}{4}$?

7. If the square of x varies directly as the cube of y , and $x = 2$ when $y = 4$, find the value of y when $x = 8$.

8. If $3x - 4$ varies directly as $y + 5$ and $x = 2$ when $y = 3$, find x when $y = 15$.

9. The volume of a sphere varies as the cube of its radius. If the volume of a sphere whose radius is 7 inches is 1437.3 cubic inches, what is the volume of a sphere whose radius is 5?

10. If $x \propto \frac{1}{\sqrt{y}}$ and $x = 4$ when $y = 36$, express the value of x in terms of y ; of y in terms of x .

11. If $x \propto y$ and $y \propto z$, show that $x \propto z$.

12. If $x \propto \frac{1}{y}$ and $y \propto \frac{1}{z}$, show that $x \propto z$.

13. The surface of a cube varies directly as the square of its edge. The surface of a cube whose edge is 6 inches is 216 square inches. What is the surface of a cube whose edge is 8 inches?

14. The safe load of a horizontal beam 10 feet long, supported at both ends, varies jointly as its breadth and the square of its depth. If a 2 by 6 pine joist of this length, standing on edge, will safely hold 800 pounds, how much will a 2 by 10 joist in the same position safely hold?

15. The amount of bending of a beam, supported at both ends and loaded at the center, varies directly as the load times the cube of the length in feet, and inversely as the width in inches times the cube of the depth in inches.

A beam 20 feet long, 5 inches wide, and 4 inches thick, supported at the ends, and loaded at the center with 4000 pounds will bend downward $2\frac{1}{4}$ inches. What must be the depth of a beam of the same material, length, and width, that shall bend only $\frac{3}{4}$ of an inch under the same load?

16. Boyle's law states that under a constant temperature the volume of a gas varies inversely as its pressure. If a gas

has a volume of 106 cubic inches under a pressure of 14.7 pounds, find the volume when the pressure is 15.3 pounds.

27. Progressions or Series. — A *series* is a succession of terms which are formed according to some fixed law. There are many kinds of series, and their treatment depends upon the nature of the law by which they are formed. In this chapter we shall consider only the two simplest series which are also called *progressions*.

28. An Arithmetic Progression is a series in which each term is formed from the preceding term by adding the same fixed number. For example, if the first term is 3 and the number added is 2, the progression is 3, 5, 7, 9, . . .

If the first term is 5 and the number added is -1 , the progression is 5, 4, 3, 2, . . .

The first term of an arithmetic progression is usually denoted by a . The number which is added to each term to produce the next term is called the *common difference*, and is denoted by d .

The general representation of an arithmetic progression is then $a, a + d, a + 2d, a + 3d, \dots$

29. A Geometric Progression is a series in which each term is formed from the preceding by multiplying by the same fixed number. For example, if the first term is 2, and the number by which we multiply is 3, the progression is 2, 6, 18, 54, . . . If the first term is 2, and the number by which we multiply is $-\frac{3}{2}$, the progression is 2, -3 , $\frac{9}{2}$, $-\frac{27}{4}$, . . .

The number by which we multiply each term to obtain the next one is called the *common ratio* and is denoted by r . If the first term is denoted by a , the general representation of a geometric progression is

$$a, ar, ar^2, ar^3, ar^4, \dots$$

Exercises

1. State the common difference in each of the following arithmetic progressions.

$$3, 4\frac{1}{2}, 6, 7\frac{1}{2}, \dots$$

$$9, 8\frac{1}{4}, 7\frac{1}{2}, 6\frac{3}{4}, \dots$$

$$x, x + y, x + 2y, \dots$$

$$3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, \dots$$

2. State the common ratio in each of the following geometric progressions.

$$3, 6, 12, 24, \dots$$

$$r, r^2, r^3, r^4, \dots$$

$$2, 2\sqrt{2}, 4, 4\sqrt{2}, \dots$$

$$10, 5, \frac{5}{2}, \frac{5}{4}, \dots$$

$$2, \sqrt{2}, 1, \frac{1}{2}\sqrt{2}, \dots$$

3. Write as an equation the statement that x , y , and z form an arithmetic progression.

4. Write as an equation the statement that x , y , and z form a geometric progression.

5. Write four terms of the arithmetic progression in which $a = 2\sqrt{2}$ and $d = -3\sqrt{2}$.

6. Write four terms of the geometric progression in which $a = \sqrt{2}$ and $r = -\sqrt{3}$.

7. Write a series of numbers beginning with 6, which may be regarded either as a special case of an arithmetic progression or as a special case of a geometric progression. State the common difference or the common ratio in each case.

8. Write four terms of the series

(a) Made by the square of the natural numbers beginning with 1.

(b) Made by squaring the odd numbers in order beginning with 1, and subtracting one from the result.

(c) Made by multiplying the number of the term in the series by the next succeeding number.

(d) In which each term is the sum of the number representing its position in the series and the square of that number.

9. Are any of the series mentioned in example 8 either arithmetic or geometric progressions?

30. Progression Formulas. — The following are important formulas of progressions.

IN ARITHMETIC PROGRESSION

a = first term.

l = last term.

n = number of terms.

d = common difference.

S = sum of n terms.

IN GEOMETRIC PROGRESSION

a = first term.

l = last term.

n = number of terms.

r = common ratio.

S = sum of n terms.

Then, $l = a + (n - 1)d$. Then, $l = ar^{n-1}$.

$$S = \frac{n}{2}(a + l).$$

$$S = \frac{ar^n - a}{r - 1} = \frac{a - ar^n}{1 - r}.$$

$$S = \frac{n}{2}[2a + (n - 1)d].$$

The derivation of these formulas is valuable but is left as a review exercise for the pupil.

The use of these formulas is shown by the following

Illustrative examples :

1. Insert four arithmetic means between 12 and -8 .

Solution : $a = 12, l = -8, n = 6.$

$$l = a + (n - 1)d.$$

$$-8 = 12 + 5d.$$

$$d = -4.$$

$$12, 8, 4, 0, -4, -8. \quad \text{Ans.}$$

2. The eighth term of an arithmetic progression is 23 and the fourth term is 11. Find the sixteenth term.

$$\begin{array}{rcl}
 \text{Solution: } a + 7d & = & 23 \\
 a + 3d & = & 11 \\
 \hline
 4d & = & 12 \\
 d & = & 3, \quad a = 2.
 \end{array}
 \qquad
 \begin{array}{rcl}
 l & = & a + (n - 1)d \\
 & = & 2 + 15 \cdot 3 = 47. \text{ Ans.}
 \end{array}$$

3. Find the sum of ten terms of the series $\frac{3}{2}, \frac{7}{4}, 2, \dots$.

$$\begin{array}{rcl}
 \text{Solution: } a & = & \frac{3}{2}, \quad n = 10, \quad d = \frac{1}{4}. \\
 S & = & \frac{n}{2} [2a + (n - 1)d] \\
 & = & 5(3 + 9 \cdot \frac{1}{4}) = 5(\frac{21}{4}) = 26\frac{1}{4}.
 \end{array}$$

4. Insert six geometric means between $\frac{27}{8}$ and $\frac{16}{81}$.

$$\begin{array}{rcl}
 \text{Solution: } l & = & ar^{n-1}. \\
 \text{Hence, } \frac{16}{81} & = & \frac{27}{8} r^7. \quad \text{Hence, } \frac{27}{8}, \frac{9}{4}, \frac{3}{2}, 1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}. \\
 r^7 & = & \frac{1 \cdot 2 \cdot 8}{2 \cdot 1 \cdot 8 \cdot 7}. \\
 r & = & \frac{2}{3}.
 \end{array}$$

5. Find the sum of six terms of the series $\frac{3}{5}, 3, 15, \dots$.

$$\begin{array}{rcl}
 \text{Solution: } a & = & \frac{3}{5}. \\
 r & = & 5. \\
 n & = & 6. \\
 S & = & \frac{ar^n - a}{r - 1} = \frac{\frac{3}{5}(5)^6 - \frac{3}{5}}{4} \\
 & = & \frac{9375 - \frac{3}{5}}{4} = \frac{11718}{5}.
 \end{array}$$

6. Find three numbers in arithmetic progression whose sum is 33 and whose product is 440.

Solution: Let the numbers be represented by $a, a + d$, and $a + 2d$. Or the equation can be made simpler by denoting the middle term by a .

Then the numbers are $a - d, a, a + d$.

Then $3a = 33$ and $a = 11$. Hence the progression is

$$a^3 - ad^2 = 440. \qquad 2, 11, 20 \text{ or } 20, 11, 2.$$

$$1331 - 11d^2 = 440.$$

$$d = \pm 9.$$

Exercises

1. Find the eighth term of the series $1, \frac{3}{4}, \frac{1}{2}, \dots$.
2. Find the twelfth term of the series $4\sqrt{3}, \sqrt{3}, -2\sqrt{3}$.
3. Find the tenth term of the series $5\sqrt{2}, \frac{14}{\sqrt{2}}, 9\sqrt{2}, \dots$.
4. Insert four arithmetic means between -62 and 3 .
5. Insert seven arithmetic means between $\sqrt{3} - \sqrt{2}$ and $5\sqrt{3} + \frac{5}{3}\sqrt{2}$.
6. Insert three arithmetic means between m and n .
7. The fifth term of an arithmetic progression is $7 - 11\sqrt{6}$, and the eighth term is $13 - 20\sqrt{6}$. Find the twelfth term.
8. The product of two numbers is 54 and their arithmetic mean is $7\frac{1}{2}$. Find them.
9. In the series $\frac{2}{3}, 1, \frac{4}{3}, \dots$, which term is $25\frac{1}{3}$?
10. Find the sum of eight terms of $4\sqrt{3}, -\sqrt{3}, -6\sqrt{3}, \dots$.
11. Find the sum of the first n odd numbers.
12. Find the sum of all numbers less than 1000 which are divisible by 4 .
13. In the series $5, 7, 9$, etc., how many terms beginning with 9 must be taken in order that their sum may be 240 ?
14. In a certain arithmetic progression whose first term is 3 , the ninth term is to the fourth term as the sixteenth term is to the seventh term. Find the series.
15. If a body falls 16 feet the first second, 48 feet the second second, 80 feet the third second, etc., how far will it fall in 6 seconds?
16. Find the eighth term of the series $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \dots$.

17. Find the sixth term of the series $2 + \sqrt{3}, 1, 2 - \sqrt{3}, \dots$.
18. Insert four geometric means between -4 and 972 .
19. Insert three geometric means between $\frac{1}{6}$ and $\frac{8}{243}$.
20. If the third term of a geometric progression is $\frac{1}{12}$ and the eighth term is $\frac{8}{3}$, find the first two terms.
21. Find the sum of six terms of the series $3, -6, 12, \dots$.
22. Find the sum of seven terms of the series $3\sqrt{2}, 12, 24\sqrt{2}, \dots$.
23. In the series $\frac{2}{3}, 2, 6, \dots$, which term is 486 ?
24. If the value of an automobile depreciates each year 40% of its value the preceding year, what is the value at the end of three years of a car which cost $\$1800$?
25. Insert two numbers between 6 and 16 such that the first three of the four form an arithmetic progression and the last three of the four form a geometric progression.
26. Prove that the arithmetic mean between a and b is $\frac{a+b}{2}$.
27. Prove that the geometric mean between a and b is $\pm\sqrt{ab}$.
28. In geometric progression derive the formula

$$S = \frac{a - rl}{1 - r}.$$

31. Infinite Geometric Progression. — The formula for the sum of a geometric progression may be written

$$S = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now if we take a geometric series in which r is less than 1 , then r^n is less than r . In fact, as n becomes larger and larger the value of r^n becomes smaller and smaller, and may be made

to approach indefinitely near to zero by taking a sufficiently great number of terms. Hence the value of ar^n , and of the fraction $\frac{ar^n}{1-r}$ also approach indefinitely near to zero. As they do so, the value of S approaches indefinitely near to $\frac{a}{1-r}$. We therefore say that the sum of an infinite number of terms, or S_∞ , $= \frac{a}{1-r}$.

You should clearly understand that the above formula gives the limit of the sum of an indefinite number of terms. If the sum of a definite number of terms is desired, we use the formula $S = \frac{a - ar^n}{1-r}$. For further discussion see authors' Second Course, page 351.

Illustrative example :

Find the sum to infinity of the series $1, \frac{1}{2}, \frac{1}{4}, \dots$.

Solution: $a = 1, r = \frac{1}{2}$.

Hence, $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

32. Repeating Decimals. — If we change a common fraction to decimal form, we shall find that the resulting decimal either comes out evenly after a certain number of figures, or else that the figures repeat in groups indefinitely.

The repeating group is called the *repetend*.

Thus,

$$\frac{1}{3} = .33333333 \dots$$

$$\frac{4}{11} = .36363636 \dots$$

$$\frac{3}{7} = .428571428571 \dots$$

$$\frac{5}{8} = .625$$

$$\frac{797}{2300} = .21515151 \dots$$

In the first example the repetend is 3, in the second 36, in the third 428571, in the fourth 3, in the fifth 15. When the

decimal repeats from the beginning, as in the first three examples, it is called a *pure repeating decimal*. When there are certain figures which do not repeat followed by repeating groups, as in the last two examples, it is called a *mixed repeating decimal*. Repeating decimals are also called *infinite decimals*, *recurring decimals*, and *circulating decimals*.

Repeating decimals furnish an excellent example of an infinite decreasing geometric progression. For example, $.3333333 \dots$ equals the sum of the terms $.3 + .03 + .003 + .0003 + .00003 + \dots$, in which the common ratio is $.1$.

$$S_{\infty} = \frac{a}{1 - r}.$$

Hence,
$$S_{\infty} = \frac{.3}{1 - .1} = \frac{.3}{.9} = \frac{3}{9} = \frac{1}{3}.$$

That is to say, the value of this decimal, if carried out indefinitely, approaches $\frac{1}{3}$ as a limit. Check by division.

Also, $.363636 \dots$ equals the sum of the series $.36 + .0036 + .000036 \dots$, in which the common ratio is $.01$.

Then,
$$S_{\infty} = \frac{.36}{1 - .01} = \frac{.36}{.99} = \frac{36}{99} = \frac{4}{11}.$$

The common ratio may be determined from the number of figures in the repetend. If one figure repeats, the ratio is $.1$; if two figures repeat, it is $.01$; if three figures repeat, it is $.001$, etc.

In the case of a mixed repeating decimal, such as the fourth above, we have $.8333333 \dots = .8 + .03 + .003 + .0003 + \dots$. Here the series begins with $.03$ and the common ratio is $.1$.

Then,
$$S_{\infty} = \frac{.03}{1 - .1} = \frac{.03}{.90} = \frac{1}{30}.$$

We now add the first term $.8$, which is not a part of the series.

$$\frac{8}{10} + \frac{1}{30} = \frac{25}{30} = \frac{5}{6}.$$

This is the sum to infinity.

Exercises

Find the sum of an infinite number of terms of each of the following series :

1. $2, \frac{4}{3}, \frac{8}{9}, \dots$ 2. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots$ 3. $5, \frac{5}{3}, \frac{5}{9}, \dots$

4. $x, 1, \frac{1}{x}, \frac{1}{x^2}, \dots$, where x is greater than 1.

5. $\frac{1}{2}, \frac{1}{4}\sqrt{2}, \frac{1}{4}, \dots$

9. $.309309\dots$

6. $.44444444\dots$

10. $.423423\dots$

7. $.272727\dots$

11. $.714825714825\dots$

8. $.848484\dots$

Find the value of each of the following :

12. $.23333\dots$

15. $.162222\dots$

13. $.566666\dots$

16. $.84515151\dots$

14. $.24545\dots$

17. If a rubber ball is dropped from a height of 4 feet, rebounds $\frac{1}{3}$ of that distance, and continues to rebound after each descent $\frac{1}{3}$ of the distance it dropped, over what distance will it have passed after 10 descents? Over what distance if it continues indefinitely?

CHAPTER III

IMAGINARY AND COMPLEX NUMBERS

IN the solution of equations and elsewhere we continually meet numbers which are neither rational nor irrational, and to which the name *imaginary numbers* has been given. It is necessary at this time that we should learn more of the nature of these numbers and understand how to use them. For purposes of distinction rational and irrational numbers are called *real numbers*.

33. An Imaginary Number is the indicated even root of a negative number ; as, $\sqrt{-2}$ and $3\sqrt{-9}$. Since the square of any real number is positive, no negative number can have a real square root. Such an expression as $\sqrt{-2}$ must, therefore, be a number different than the real numbers we have studied.

34. A Complex Number is the indicated sum or difference of a real number and an imaginary number ; as, $3 + \sqrt{-1}$, $-2 - 5\sqrt{-3}$.

A complex number is, of course, imaginary. In distinction from the complex number, such an expression as $-3\sqrt{-1}$ is often called a *pure imaginary*.

35. Simplification of Imaginaries. — An imaginary number must of necessity be a radical, and many of the principles of simplification which we have studied under real radicals apply equally to imaginaries.

Illustrative examples :

Simplify $-2\sqrt{-9}$; $3\sqrt{-\frac{2}{3}}$.

Solutions: $-2\sqrt{-9} = -2\sqrt{9(-1)} = -6\sqrt{-1}$.

$$3\sqrt{-\frac{2}{3}} = 3\sqrt{-\frac{2}{3} \cdot \frac{3}{3}} = 3\sqrt{\frac{1}{9}(-6)} = \sqrt{-6}.$$

Since $\sqrt{-6} = \sqrt{6(-1)}$ this may also be written in the form of $\sqrt{6}\sqrt{-1}$.

36. The Imaginary Unit.—The illustrative examples above show that every imaginary number can be written as a real number times $\sqrt{-1}$. Hence, $\sqrt{-1}$ is considered the unit of the system of imaginary numbers just as 1 is the unit of the system of real numbers.

The number $\sqrt{-1}$ is often called “*i*,” because it is the initial letter of the word *imaginary*.

Thus, $\sqrt{-9} = 3\sqrt{-1}$ or $3i$.

$$2\sqrt{-3} = 2\sqrt{3}\sqrt{-1} = 2i\sqrt{3}, \text{ etc.}$$

The imaginary unit *i* is used like the real terms or factors which you found in the beginning of your algebra study.

Thus, $4 = +1 + 1 + 1 + 1$.

And so $4i = i + i + i + i$.

$$-4 = -1 - 1 - 1 - 1.$$

And so $-4i = -i - i - i - i$, etc.

We then have two sets of imaginary numbers just as we had two sets of real numbers in beginning algebra.

Thus, $-4i, -3i, -2i, -i, 0, +i, +2i, +3i, +4i, \dots$

The imaginary number then is imaginary in the same sense that a negative number or an irrational number is imaginary to a person knowing only the positive integers.

37. General Representation of a Complex Number. — Any complex number may be represented in the form $a + bi$ when a and b represent any real numbers whatever. If b has the value 0 while a takes on all possible real values, we have the system of real numbers resulting. If a has the value 0 while b takes on all possible real values, we have the system of pure imaginaries resulting. It follows that both real numbers and pure imaginaries may be regarded as special cases of the complex number.

Illustrative example I:

Simplify and combine $2\sqrt{-27} + 5\sqrt{-75} - 3\sqrt{-12}$.

Solution:

$$\begin{aligned} 2\sqrt{-27} + 5\sqrt{-75} - 3\sqrt{-12} \\ &= 2\sqrt{9(-3)} + 5\sqrt{25(-3)} - 3\sqrt{4(-3)} \\ &= 6\sqrt{-3} + 25\sqrt{-3} - 6\sqrt{-3} \\ &= 25\sqrt{-3}. \end{aligned}$$

Illustrative example II:

Simplify and combine $\sqrt{-\frac{1}{2}} - 3\sqrt{-\frac{1}{18}} + 2\sqrt{-\frac{25}{8}}$.

Solution:

$$\begin{aligned} \sqrt{-\frac{1}{2}} - 3\sqrt{-\frac{1}{18}} + 2\sqrt{-\frac{25}{8}} \\ &= \sqrt{-\frac{1}{2} \cdot \frac{2}{2}} - 3\sqrt{-\frac{1}{18} \cdot \frac{2}{2}} + 2\sqrt{-\frac{25}{8} \cdot \frac{2}{2}} \\ &= \frac{1}{2}\sqrt{-2} - \frac{1}{2}\sqrt{-2} + \frac{5}{2}\sqrt{-2} \\ &= \frac{5}{2}\sqrt{-2}. \end{aligned}$$

Exercises

Simplify and combine

- $2\sqrt{-8} - 3\sqrt{-8} + 4\sqrt{-72}$.
- $\frac{1}{2}\sqrt{-9} + 6\sqrt{-4} + \sqrt{-25}$.
- $4\sqrt{-20} - 6\sqrt{-80} + 2\sqrt{-45}$.

4. $a\sqrt{-a^2x} - 2b\sqrt{-b^2x} - 5c\sqrt{-c^2x}.$
5. $2\sqrt{-a^2 - b^2 + 2ab} + 3\sqrt{-6ab - a^2 - 9b^2}.$
6. $2\sqrt{-\frac{1}{2}} + \sqrt{-50} + 6\sqrt{-200} + 5\sqrt{-\frac{9}{2}}.$
7. $\sqrt{-12} + 4\sqrt{-75} - \sqrt{-\frac{100}{3}}.$
8. $3\sqrt{-x^2} + 4x\sqrt{-9} - \frac{2}{x}\sqrt{-25x^4}.$
9. $2\sqrt{-12} + \frac{3}{2}\sqrt{-48} - 6\sqrt{-\frac{1}{3}}.$
10. $3\sqrt{-50} - \frac{5}{2}\sqrt{-\frac{3}{2}} + 6\sqrt{-\frac{2}{3}} - \sqrt{-\frac{32}{9}}.$

Review

1. The sum of the means of a proportion is 7, the sum of the extremes is 8, and the sum of the squares of all the terms is 65. What is the proportion?

2. Divide $(x^2 + 1)$ by $(x^{\frac{2}{3}} + 1)$.

3. Show whether the following expression is a perfect square or not:

$$e^{4x} + 2e^{3x} + e^{2x} + 2e^x + 2 + e^{-2x}.$$

38. Addition and Subtraction of Complex Numbers. — Complex numbers in the form $a + bi$ may be added or subtracted as in the case of real numbers.

Illustrative example I:

$$\begin{array}{rcl} \text{Add,} & & 2 + 5\sqrt{-1} \\ & & 3 - 8\sqrt{-1} \\ \hline \text{Sum =} & & 5 - 3\sqrt{-1} \end{array}$$

Illustrative example II:

$$\begin{array}{rcl} \text{Add,} & & a + b\sqrt{-1} \\ & & c + d\sqrt{-1} \\ \hline \text{Sum =} & & (a + c) + (b + d)\sqrt{-1} \end{array}$$

Illustrative example III :

$$\begin{array}{rcl}
 \text{Subtract,} & & 3 - 6i \\
 & & \underline{2 + 5i} \\
 \text{Difference =} & & 1 - 11i.
 \end{array}$$

Exercises

Perform the indicated operations.

1. $(3 + 2\sqrt{-1}) + (2 + 3\sqrt{-1})$.
2. $(2 + i) + (-3 + i)$. 4. $(1 + 2i) + (-3 + 3i)$.
3. $(1 + i) - (-2 + i)$. 5. $(-3 + 4i) - (-2 - 2i)$.
6. $(a + bi) + (c + 2di)$.
7. $(4 + \sqrt{-4}) + (-5 - \sqrt{-16})$.
8. $(5 - \sqrt{-4}) + (-4 - \sqrt{-9})$.
9. $(-2 + \sqrt{-3}) - (-2 - \sqrt{-3})$.
10. $(\frac{1}{4} - \frac{1}{4}i\sqrt{3}) + (3 + 3i\sqrt{3})$.
11. $(22 - \sqrt{-12}) + (2 - \sqrt{-3})$.

Review

1. Find three numbers in geometric progression such that their sum shall be 104 and the last shall exceed the first by 64.
2. Compute the larger of the two roots of the equation

$$x^2 = 0.100 - 0.200x$$
 correct to three significant figures.

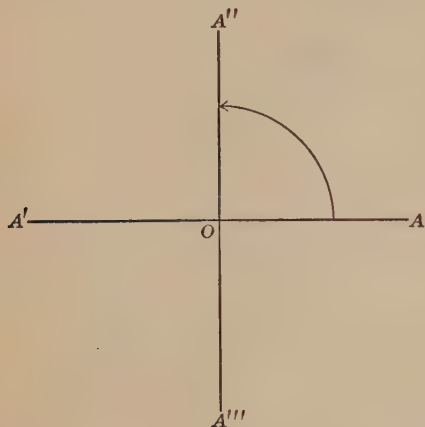
3. Factor

$$(a) \quad 8c^2x - 12c^2y - 2d^2x + 3d^2y.$$

$$(b) \quad x^2 - y^2 + c^2 - d^2 - 2cx + 2dy.$$

39. Graphic Representation of Complex Numbers. — We shall now develop a method by which complex numbers may

be represented on a plane. Let us suppose that from the point 0, a distance OA be taken equal to 1 unit. Then an equal length OA' is taken in the opposite direction from 0. The point A then may represent $+1$ and the point A' represents -1 , while 0 represents zero. Using this as a



basis we may represent on the line AA' extended indefinitely every real number whatever.

Now the line OA' may be considered as obtained by rotating OA through 180° in the direction indicated by the arrow. The same operation repeated on OA' again produces OA . Hence these

equal rotations may be regarded as in effect producing two equal multiplications. Thus,

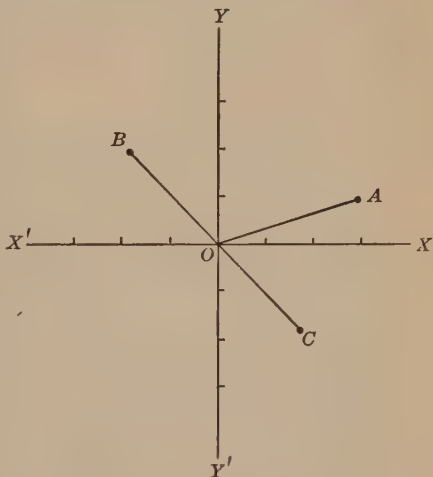
$$1(-1) = -1; (-1)(-1) = +1.$$

In like manner the rotation from OA to OA' consists of two equal rotations through 90° . In order that we may be consistent with our previous discussion we shall consider that each of these represents a multiplication which twice performed produces -1 . Then OA'' must equal $\sqrt{-1}$.

Since the above discussion shows that rotation through 90° is equivalent to multiplying by $\sqrt{-1}$, it follows that $OA''' = -1 \cdot \sqrt{-1} = -\sqrt{-1}$. Hence by extending the axis $A''A'''$ indefinitely we may locate on it points which represent any pure imaginary.

The graph shows that 0 may be regarded as belonging to both the system of real numbers and the system of imaginary numbers, since it is on both axes. This is obvious also from the algebraic standpoint when we consider that 0 may be regarded either as $0 \cdot 1$ or $0 i$.

In the diagram which follows, we shall call xx' the real axis and yy' the imaginary axis. Then a complex number such as $3 + i$ may be represented by locating a point 3 units to the right and one unit up, as the point A . In a similar manner $-2 + 2i$ will be located two points to the left and 2 units up as the point B , and the number $2 - \sqrt{-3}$, which equals $2 - \sqrt{3}i$, will be located 2 units to the right and approximately 1.732 units down as the point C .



It follows that every complex number may be represented as a point on the plane determined by the axes xx' and yy' and conversely every point in that plane represents some complex number. This plane is spoken of as the *complex plane*.

Exercises

Represent on a diagram the following points.

- | | | | |
|--------------|----------------|----------------|--------------------------|
| 1. $4 + i$. | 3. $5 - 2i$. | 5. $-3 + 2i$. | 7. $3 + 1.5i$. |
| 2. $4 - i$. | 4. $-2 - 3i$. | 6. $-4 + 3i$. | 8. $-2 + \frac{5}{2}i$. |

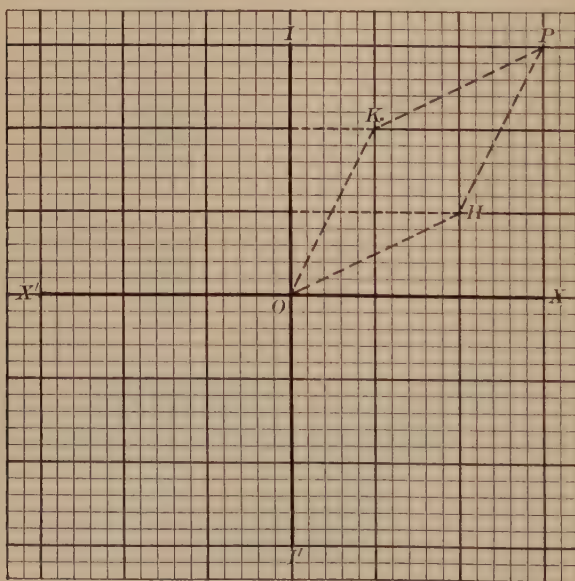
9. $3i + 2$. 10. $-3i - 3$. 11. $4 + 2i$. 12. $4 - 2i$.

13. Represent the sum of $2 + i$ and $2 - i$.

14. The vertices of a triangle are $3 + 2i$, $-2 + 2i$, and $-1 - 3i$. Construct the triangle.

15. The vertices of a certain plane figure are $5 + 4i$, $-6 + 4i$, $5 - 3i$, and $-6 - 3i$. Plot these points. What is the figure? What is its area? What is the area of the figure whose vertices are the last three mentioned?

40. Graphic Addition of Two Complex Numbers. — The addition of two complex numbers may be graphically illustrated by the following:



Illustrative example I:

Add graphically $1 + 2i$ and $2 + i$.

Solution: First construct on the chart the point H , $(2 + i)$, then construct the point K , $(1 + 2i)$.

Construct OH and OK , then construct the parallelogram $OHPK$. We have now located the point P . Point P represents $(1 + 2i) + (2 + i)$. The result is $3 + 3i$.

If we were to have drawn HP , equal and parallel to OK , we could have easily found the point P . This is due to a fact which you proved in your geometry. What was it?

If preferred, a second method, sometimes called the point method, of adding complex numbers may be used, as shown in

Illustrative example II:

Solve the equation $x^3 - 1 = 0$ and show graphically that the sum of the roots is 0.

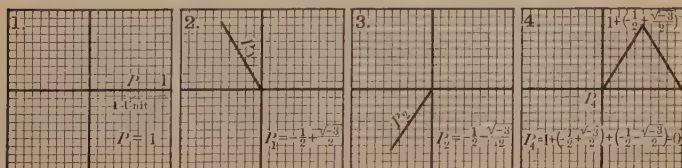
Solution:

$$x^3 - 1 = 0.$$

$$(x - 1)(x^2 + x + 1) = 0.$$

$$x = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

If you will plot these values as illustrated, taking them in the order of the charts, you will observe that the length and direction of the line shown in chart 3 will just reach 0 in chart 4. Care must be taken in the matter of direction and lengths in the construction of the lines in chart 4 so that they shall correspond to those of charts 1, 2, and 3 in that particular order.

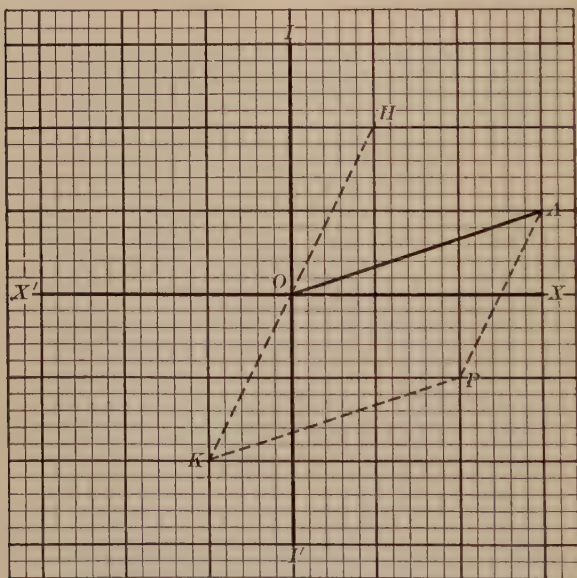


Exercises

Represent graphically:

- | | |
|----------------------------|----------------------------|
| 1. $(1 + 3i) + (3 + i)$. | 5. $(-2 + i) + (+2 + i)$. |
| 2. $(1 + 2i) + (2 - 3i)$. | 6. $2 + (3 - i)$. |
| 3. $(2 + 3i) + (3 - 2i)$. | 7. $(-4i) + (3 + i)$. |
| 4. $(3 + 2i) + (3 - 2i)$. | 8. $(4 - 2i) + (-5i)$. |

41. Graphical Representation of the Subtraction of Two Complex Numbers. — To subtract $1 + 2i$ from $3 + i$ is equivalent to adding $-1 - 2i$ to $3 + i$.



In order to do this graphically we find the point K , which represents $-1 - 2i$. Then we find the point A , which represents $3 + i$.



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Next we construct from A , a line equal and parallel to OK . The point P , shown in the adjoining figure, represents the result sought, that is, $2 - i$.

Exercises

Represent graphically:

1. $(1 + 3i) - (3 + i)$.

3. $(2 + 3i) - (3 - 2i)$.

2. $(1 + 2i) - (2 - 3i)$.

4. $(-4i) - (3 + i)$.

HISTORICAL NOTE

Just as negative roots were not understood for many, many years, in the same way, the imaginary number and complex number were not understood by mathematicians for a long time. When imaginaries began to appear as roots of an equation, it was then, and then only, that some idea of their meaning began to be gained by mathematical students.

They were not well understood until their graphical representation was shown by some writers. It was not until about 1800 that a Frenchman, Argand, and a Norwegian, Wessel, gave about the same representation of them. Karl Friedrich Gauss (1777–1855) about 1800 published his demonstration that every algebraic equation has a root of the form of $a + bi$. He first put the subject upon a systematic and scientific basis.

42. Powers of i . — The table which follows gives the powers of i in succession.

$$i = \sqrt{-1}.$$

$$i^2 = (\sqrt{-1})^2 = -1.$$

$$i^3 = i^2 \cdot i = (\sqrt{-1})^2 \sqrt{-1} = -1 \sqrt{-1}.$$

$$i^4 = (i^2)^2 = (-1)^2 = +1.$$

$$i^5 = i^4 \cdot i = \sqrt{-1}.$$

$$i^6 = i^4 \cdot i^2 = -1, \text{ etc.}$$

You will observe that i^5 is the same, in value, as i ; i^6 the same as i^2 , etc. All powers of i reduce to one of the four quantities $\sqrt{-1}$, -1 , $-\sqrt{-1}$, and $+1$.

When the powers are arranged in order, they produce an indefinite number of cycles of these four quantities in the order given.

Thus, $i^{46} = i^2 = -1$ (since $46 = 11 \cdot 4 + 2$).

And $i^{99} = i^3 = -\sqrt{-1}$ (since $99 = 24 \cdot 4 + 3$).

43. Multiplication of Imaginaries. — Let it be required to multiply $3\sqrt{-4}$ by $2\sqrt{-6}$.

$$\text{Solution: } 3\sqrt{-4} = 6\sqrt{-1}$$

$$2\sqrt{-6} = 2\sqrt{6}\sqrt{-1}$$

$$\text{The product} = 12\sqrt{6}(-1) = -12\sqrt{6}$$

The product of two pure imaginary numbers is always real.

Again multiply $2 - 3\sqrt{5}i$ by $1 + 2\sqrt{-5}$.

$$\begin{array}{r} \text{Solution:} \quad 2 - 3\sqrt{5}i \\ \quad 1 + 2\sqrt{5}i \\ \hline 2 - 3\sqrt{5}i \\ \quad + 4\sqrt{5}i - 30i^2 \\ \hline 2 + \sqrt{5}i - 30i^2 = 32 + \sqrt{5}\sqrt{-1} \end{array}$$

Conjugate complex numbers are those which differ only in the sign of the imaginary term, as $3 - 2\sqrt{-6}$ and $3 + 2\sqrt{-6}$.

The product of two conjugate complex numbers is real.

$$\begin{array}{r} \text{Thus,} \quad 3 - 2\sqrt{6}i \\ \quad 3 + 2\sqrt{6}i \\ \hline 9 - 6\sqrt{6}i \\ \quad + 6\sqrt{6}i - 24i^2 \\ \hline 9 - 24i^2 = 9 + 24 = 33 \end{array}$$

Note the similarity to $(a - b)(a + b)$.

In general, $(a + ib)(a - ib) = a^2 + b^2$,

and $(a + ib)(c + id) = ac - bd + i(cb + ad)$.

44. Division of Imaginaries. — In dividing by an imaginary or complex number we indicate the division in fractional form, and simplify as in principle 5 (authors' Second Course, page 178) of real radicals.

Illustrative example I:

Divide 3 by $2\sqrt{-3}$.

Solution:

$$\begin{aligned}\frac{3}{2\sqrt{-3}} &= \frac{3}{2\sqrt{3}\sqrt{-1}} \cdot \frac{\sqrt{3}\sqrt{-1}}{\sqrt{3}\sqrt{-1}} = \frac{3\sqrt{3}\sqrt{-1}}{-6} \\ &= -\frac{\sqrt{3}\sqrt{-1}}{2}.\end{aligned}$$

Illustrative example II:

Simplify $\frac{3 + \sqrt{-5}}{3 - \sqrt{-5}}$.

Solution:

$$\begin{aligned}\frac{3 + \sqrt{-5}}{3 - \sqrt{-5}} &= \frac{3 + \sqrt{-5}}{3 - \sqrt{-5}} \cdot \frac{3 + \sqrt{-5}}{3 + \sqrt{-5}} = \frac{4 + 6\sqrt{-5}}{14} \\ &= \frac{2 + 3\sqrt{-5}}{7}.\end{aligned}$$

Exercises

Perform the indicated operations and simplify the results:

1. $\frac{4}{3\sqrt{-1}}$

3. $\frac{1 + i}{3i^3}$

2. $\frac{5}{i}$

4. $\frac{2 + 3\sqrt{-2}}{2\sqrt{-2}}$

5. $(2 + 5\sqrt{-1})(3 - 2\sqrt{-1})$.
6. $(2 + 3\sqrt{-1} - 2\sqrt{-2})(1 + 3\sqrt{-1} + 4\sqrt{-2})$.
7. $(2 + \sqrt{-1} - \sqrt{-3})(2 - \sqrt{-1} + \sqrt{-3})$.
8. $(1 + i)^3$.
9. $(1 - i)^3$.
10. $(-\frac{1}{2} + \frac{1}{2}\sqrt{-3})^3$.
11. $\frac{2\sqrt{-1}}{3 + 2\sqrt{-1}}$.
12. $\frac{3\sqrt{-2}}{1 - \sqrt{-1}}$.
13. $\frac{2 + \sqrt{-3}}{2 - \sqrt{-3}}$.
14. $\frac{3 - \sqrt{-5}}{2 + 2\sqrt{-5}}$.
15. $\frac{-2 + \sqrt{-4}}{-1 + \sqrt{-4}}$.
16. $6\sqrt{-3} \cdot 4\sqrt{-3}$.
17. $(a + \sqrt{-b})(a - \sqrt{-b})$.
18. $(3 - \sqrt{-1})(3 + \sqrt{-1})$.
19. $(3 - \sqrt{-2})^2$.

Divide:

20. $\sqrt{24}$ by $-\sqrt{-6}$.
21. $\sqrt{-12}$ by $\sqrt{-3}$.
22. $\sqrt{-45}$ by $2\sqrt{-9}$.
23. $\sqrt{-125}$ by $\sqrt{-25}$.
24. $(2 + 5\sqrt{-1})$ by $(2 - 5\sqrt{-1})$.
25. $(a - b\sqrt{-1})$ by $(a + b\sqrt{-1})$.
26. $(x + yi)$ by $(x + 2yi)$.
27. Is $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ a root of $x^2 + x + 1 = 0$?
28. Simplify $(1 - i)^3 - 2(1 - i)^2 + 5(1 - i)$.
29. If $x = \frac{1 - i}{2}$, compute the value of $x^2 - 3x + 4$.

30. Solve the equation $x^2 + 2x + 2 = 0$. Represent the roots graphically. Represent the sum of the roots graphically.

31. Express with a real denominator

$$\frac{3\sqrt{-2} + 2\sqrt{-5}}{3\sqrt{-2} - 2\sqrt{-5}}$$

Simplify and express in the form of $a + bi$.

$$32. \frac{3 + 5i}{2 - 3i}.$$

$$33. \frac{3 + \sqrt{-1}}{3 - \sqrt{-1}}.$$

Find complex factors of each of the following :

$$34. x^2 + 4x + 5.$$

$$36. 2x^2 + 3x + 4.$$

$$35. x^2 - 8x + 21.$$

$$37. x^2 + 6x + 10.$$

38. Solve the equation $x^3 - 1 = 0$. Represent the roots graphically. Represent their sum graphically.

39. Solve the equation $x^3 + 1 = 0$. Represent the roots graphically. Represent their sum graphically.

Review

Find the square root of each of the following :

$$(a) 9 - 2\sqrt{14}.$$

$$(e) 10 - 4\sqrt{6}.$$

$$(b) 14 + 2\sqrt{33}.$$

$$(f) 4\frac{1}{3} - \frac{4}{3}\sqrt{3}.$$

$$(c) 7 - \sqrt{48}.$$

$$(g) 3x + 2x\sqrt{2}.$$

$$(d) 24 + 3\sqrt{28}.$$

$$(h) 111 - 2\sqrt{3024}.$$

45. Square Root of a Complex Number. — The square root of a complex number may be obtained by a method analogous to that used in finding the square root of a binomial surd. (See Second Course in Algebra, page 186.) The method is sufficiently explained in the following

Illustrative example :

Find the square root of $-5 - 12\sqrt{-1}$.

Solution: Writing this expression so that the coefficient of the radical is 2, we have,

$$-5 - 2\sqrt{-36}.$$

Find, by trial, two numbers whose product is -36 and whose sum is -5 . We obtain -9 and $+4$.

$$\text{Hence, } \sqrt{-5 - 2\sqrt{-36}} = \sqrt{4} - \sqrt{-9} = 2 - 3\sqrt{-1}.$$

46. Cube Roots of Unity. — The three cube roots of 1 may be found algebraically by solving the equation $x^3 = 1$.

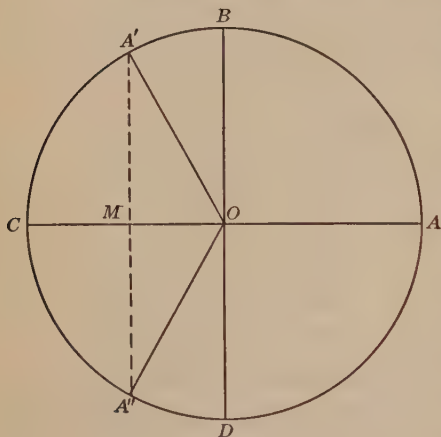
Solution: $x^3 - 1 = 0.$

$$(x - 1)(x^2 + x + 1) = 0.$$

$$x = 1. \quad x = -\frac{1 \pm \sqrt{-3}}{2}.$$

The imaginary cube roots of 1, $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$ are

known in mathematics as ω (Omega) and ω^2 respectively. They have the following remarkable properties. Either is the square of the other. Hence either is also the square root of the other. It follows that the square and square root of either are the same quantity.



Their product is 1, since it is the cube of either.

The three cube roots of 1 may be represented graphically by the points A , A' , and A'' . Since $OM = \frac{1}{2}$ (in absolute value) and $MA' = \frac{1}{2}\sqrt{3}$, it follows by geometry that $OA' = 1$. Similarly, $OA'' = 1$. Also, $\angle A'OM = 60^\circ$. Hence, $\angle AOA' = \angle A'OA'' = \angle A''OA = 120^\circ$. This is exactly in accord with our discussion in section 39, for it was there shown that we could extract the square root of a number by dividing the angle of rotation which produces that number into two equal parts. Hence it is reasonable to suppose that we should obtain the cube roots of 1 by dividing the $\angle 360^\circ$

or a complete revolution which produces $+1$, into 3 equal parts.

Hence: *We may represent graphically the n th roots of any positive number by drawing a circle with origin as the center and the real n th root of that number as a radius, taking the point where this circle cuts the positive side of the real axis as one root, and beginning at that point and proceeding in the positive direction of rotation, dividing the circle into n equal parts.*

Exercises

Obtain algebraically the square roots of each of the following complex numbers.

1. $4 - 2\sqrt{-21}$.

5. $9 + 4\sqrt{-\frac{11}{2}}$.

2. $-5 + 12\sqrt{-1}$.

6. $8 + 6\sqrt{-1}$.

3. $-6 + 12\sqrt{-6}$.

7. $-9 + 40\sqrt{-1}$.

4. $4 + 6\sqrt{-5}$.

8. $16 - 30\sqrt{-1}$.

Represent graphically:

9. The 4 fourth roots of 1. 11. The 5 fifth roots of 32.

10. The 3 cube roots of 8. 12. The 6 sixth roots of 729.

47. Theorem: *If a complex number equals 0, the real part is 0 and the imaginary part is 0, separately.*

Proof:

Let $a + bi = 0$.

Then, $a = -bi$,

and, squaring, $a^2 = -b^2$.

But a^2 and b^2 are essentially positive since they are squares of real numbers.

We have here the statement that a positive number is equal to a negative number, which can be true only if both are 0, for $+0 = -0$.

Hence, $a^2 = 0$ and $b^2 = 0$.

That is, $a = b = 0$.

Corollary. — *If two complex numbers are equal to each other, the real parts are equal, and the imaginary parts are equal, separately.*

Proof:

Let $a + bi = c + di$.

Then, $(a - c) + (b - d)i = 0$.

Hence by the preceding theorem,

$$a - c = 0 \text{ and } b - d = 0.$$

Therefore, $a = c$ and $b = d$.

CHAPTER IV

INEQUALITIES

IN expressing mathematical relations it is often necessary to state that two magnitudes are unequal. The laws governing such statements, together with their meaning and graphic illustration, constitute the subject matter of the present chapter.

48. An Inequality is a statement that two magnitudes are unequal. It may be expressed by any one of three signs, $\neq$ (is not equal to), $>$ (is greater than), or $<$ (is less than). The first of these is less definite than the others since it does not say which of the two magnitudes is the greater.

49. The Sense of the Inequality is determined by the sign used, that is $>$ or $<$. If two are expressed by the same sign, they are said to be "in the same sense," while otherwise they are in the "opposite sense."

50. Axioms of Inequality. — The treatment of inequalities, as in the case of equations, is based upon certain fundamental principles or axioms.

1. *If equals are added to unequals, the sums are unequal in the same sense.*

2. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

NOTE. It follows that we may *transpose* in an inequality, just as we do in an equation.

3. *If both members of an inequality are multiplied or divided by positive equals, the result is an inequality in the same sense.*

4. *If the signs of both members of an inequality are changed, the inequality sign must be reversed.*

5. *If both members of an inequality are multiplied or divided by negative equals, the result is an inequality in the opposite sense.*

This is equivalent to multiplying or dividing by positive equals, and then changing the signs of both members.

6. *Like powers or like roots of positive unequals are unequal in the same sense.*

7. *A series of positive inequalities in the same sense may be either added or multiplied without affecting the sense of the inequality.*

Just as equations may be either *identical* or *conditional*, inequalities may be either *absolute* or *conditional*.

51. An Absolute Inequality is one which is true unconditionally for the range of values considered.

Thus, if a is a positive number, $2a > a$. Numerical inequalities are always absolute. For example, $10 > 7$, $3 < 12$.
 $-4 > -10$.

Much of the work of absolute inequalities is based on the following

Theorem: *The sum of the squares of any two unequal real numbers is greater than twice their product. That is, if $a \neq b$, $a^2 + b^2 > 2ab$.*

Proof:

Let $a \neq b$.

Then, $a - b \neq 0$.

Squaring, $a^2 - 2ab + b^2 > 0$.

Therefore, $a^2 + b^2 > 2ab$.

This statement is often called the *fundamental inequality*.

The following examples illustrate two methods of treating inequalities.

Illustrative example I:

Prove that $\frac{a+b}{2} > \frac{2ab}{a+b}$ if $a \neq b$, and $a+b > 0$.

This statement is true if

$$a^2 + 2ab + b^2 > 4ab,$$

which is true if

$$a^2 + b^2 > 2ab.$$

But this is true by the fundamental inequality,

Hence,
$$\frac{a+b}{2} > \frac{2ab}{a+b}.$$

Illustrative example II:

Prove $a^3 + b^3 > a^2b + ab^2$ if $a \neq b$, $a > 0$, $b > 0$.

This statement is true if

$$a^3 - a^2b - ab^2 + b^3 > 0.$$

If
$$a^2(a-b) - b^2(a-b) > 0.$$

If
$$(a^2 - b^2)(a-b) > 0.$$

If
$$(a+b)(a-b)^2 > 0.$$

But since a and b represent unequal and positive numbers, then $a+b$ is positive and $(a-b)^2$ is positive.

Hence,
$$(a+b)(a-b)^2 > 0.$$

It follows that
$$a^3 + b^3 > a^2b + ab^2.$$

Exercises

Except as otherwise stated, it is understood that the letters in the following examples represent *unequal* and *positive real* numbers.

Show that

1. $a^4 + b^4 > a^3b + ab^3.$

2. $(ab + cd)(ac + bd) > 4abcd.$

3. $(x + y)(y + z)(x + z) > 8xyz$.
4. The arithmetic mean between two positive numbers is greater than their geometric mean.
5. $3xy^2 < x^3 + 2y^3$.
6. $x^2 + y^2 + z^2 > xy + yz + xz$.
7. If $m^2 + n^2 = 1$ and $p^2 + q^2 = 1$, prove that $mp + nq < 1$.
8. If $x \neq 1$, prove that $x + \frac{1}{x} > 2$.
9. Show that $a^3 + b^3 + c^3 > 3abc$.
10. Which is greater $2x^3 + 3y^3$ or $4xy^2 + x^2y$?
11. Which is greater $\frac{m^2 + n^2}{m + n}$ or $\frac{m^3 + n^3}{m^2 + n^2}$?
12. If $a \neq 3b$, prove that $b^2 > (a - 2b)(4b - a)$.
13. Show that $\sqrt{7} + \sqrt{5} < 2\sqrt{6}$.
14. Prove that $3(a^2 + b^2 + c^2) > (a + b + c)^2$.

Review

1. Simplify

$$\sqrt{-10} \cdot \sqrt{-5} \cdot \sqrt{-2} \cdot \sqrt{-1}.$$

2. Simplify $\left(\frac{x^{2n}}{x^{n+1}}\right)^{n-1} \div \left(\frac{x^{n-1}}{x^n}\right)^{n+1}$.

3. Solve $3x - \frac{6x^2 - 40}{2x - 1} - \frac{3x - 10}{9 - 2x} = 2$.

4. Solve $x^2 + 5x + 4 = 5\sqrt{x^2 + 5x + 28}$.

5. Show that $\left(\frac{-1 + \sqrt{-3}}{2}\right)^3 + \left(\frac{-1 - \sqrt{-3}}{2}\right)^3 = 2$.

6. Simplify

$$\frac{1}{\sqrt{1 - \frac{x^2}{1+x^2}}} \cdot \frac{\sqrt{1+x^2} - x^2(1+x^2)^{-\frac{1}{2}}}{(\sqrt{1+x^2})^2}.$$

7. Solve

$$\begin{aligned} 2x - 3y &= 6. \\ 4y - 5z &= 7. \\ 2z - 3w &= 8. \\ -5x &+ 4w = 9. \end{aligned}$$

52. Conditional Inequalities are true only for certain values of the letters concerned. For example, $5x > 10$ is true only if $x > 2$. The value thus obtained is called the *limit* of x in the inequality, and the process of obtaining it is called solving the inequality. Inequalities are solved by the use of the axioms above listed.

Illustrative example I:

Solve $3x - 2 < 4x + 6.$

Solution: Transposing, we obtain,

$$-x < 8.$$

$$\therefore x > -8. \quad \text{Axiom 4.}$$

Illustrative example II:

Solve $x^2 + x < 6.$

Solution: $x^2 + x < 6.$

Then, $x^2 + x - 6 < 0.$

$$(x+3)(x-2) < 0.$$

A product of two factors can be less than 0, only when one factor is positive and the other negative.

Hence, $x+3 > 0$ and $x-2 < 0$, in which case
 $x > -3$ and $x < 2.$

Or else $x+3 < 0$ and $x-2 > 0$, in which case
 $x < -3$ and $x > 2.$

No values of x exist which are < -3 and at the same time are > 2 . Hence the solution is $x > -3$ and < 2 . That is to say, the inequality is true for all values of x lying between -3 and 2 . In other words, -3 and 2 are the limits of the values of x .

Illustrative example III :

What are the limits of the values of x , if

$$x - 5 < 4 - 2x, \quad (1)$$

and $5 - 2x > 7 - 4x ? \quad (2)$

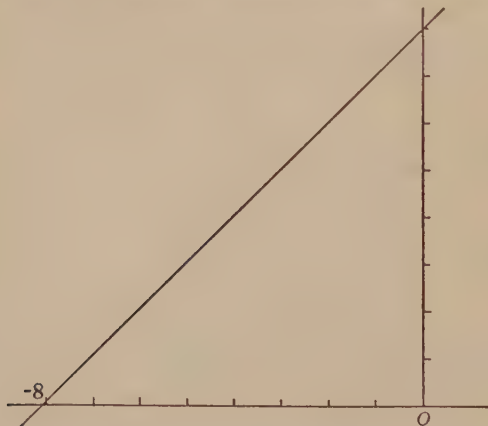
Solution:

Transposing in (1), $3x < 9$ or $x < 3$.

Transposing in (2), $2x > 2$ or $x > 1$.

Hence the values of x lie between $+3$ and $+1$.

53. Graphic Illustrations. — The following examples illustrate the graphic treatment of conditional inequalities.



Let us consider the inequality discussed in the first illustrative example above, $3x - 2 < 4x + 6$.

This may be written $-x - 8 < 0$ or $x + 8 > 0$.

The linear function $x + 8$ is represented graphically on page 74.

This function is positive when it is above the X -axis.

We see from the graph that those values of x which make the line lie above the X -axis are $x > -8$.

Similarly,

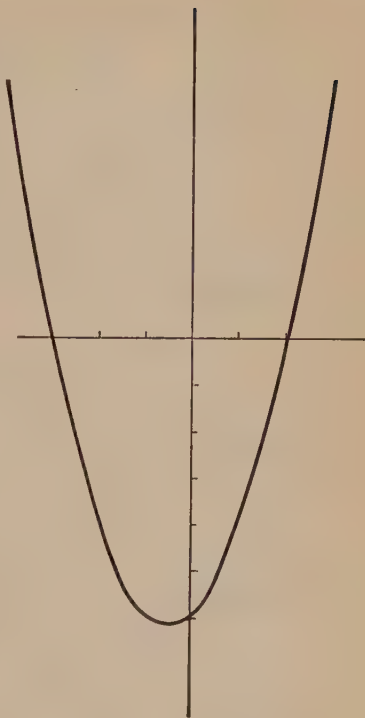
$$x^2 + x < 6,$$

may be written

$$x^2 + x - 6 < 0.$$

The graph of a quadratic function of x is a parabola. The graph of $x^2 + x - 6$ gives us the figure on this page. $x^2 + x - 6 < 0$ for all values of x lying between -3 and $+2$.

Compare with the algebraic solution.



Exercises

Solve both algebraically and graphically :

1. $3x - 2 > x + 6.$

4. $\frac{2x - 1}{3} < x - \frac{x - 1}{2}.$

2. $\frac{x}{2} < \frac{3x}{4} + 1.$

5. $x^2 < 2x.$

3. $\frac{2x}{3} - x > 3x + \frac{1}{2}.$

6. $x^2 - 3x > 10.$

7. $x^2 + 3 > 4x.$

8. $1 > 2x^2 + x.$

9. Find the limits of the values of x in

$$6x + 1 > 0,$$

and

$$25 - 4x > 0.$$

10. Find the limits of the values of x and y in

$$7x + y = 15,$$

and

$$3x - 2y > 14.$$

Hint: Solve algebraically using the idea of elimination as in simultaneous equations.

Cumulative Review

1. Simplify $\frac{\sqrt{6} - \sqrt{-12}}{\sqrt{6} + \sqrt{-3}}.$

2. Simplify $\frac{6 - 2\sqrt{-3}}{7 + \sqrt{-27}}.$

3. Show that $x = 9\sqrt{2} - 11$ is a solution of the equation

$$\frac{(\sqrt{2} + 1)x}{5 - 9\sqrt{2}} = \frac{2\sqrt{2} - 7}{x + 6}.$$

4. Reduce $(3 - x)^2 > (x - 4)^2.$

5. Find the product of $a + b$, $a - b$, $a + b\sqrt{-1}$, and $a - b\sqrt{-1}.$

6. Multiply $3 - 2i$ by $2 + i$ and subtract graphically the second factor from the product.

7. What rotation is equivalent to the multiplication by $-i$?

8. The *modulus* or *absolute value* of any number is the *length of the line* from the origin to the point which represents that number. If the absolute value of $a + bi = +\sqrt{a^2 + b^2}$, what is the absolute value of $-3 - 4i$?

9. Solve the following inequality, $x^2 - 225 < 0.$

10. Factor :

(a) $6x^2 + 59xy + 105y^2$. (c) $a^4 + 16 + 4a^2$.

(b) $6x^2 - 7x - 20$. (d) $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24$.

11. Find the least multiple of 7 which, when divided by 2, 3, 4, 5, 6, leaves in each case 1 for a remainder.

12. Find the H. C. F. of $a^3 + 1$, $3a^3 - 4a^2 + 4a - 1$, $2a^3 + a^2 - a + 3$.

13. Given $H = \frac{v}{825} \left(T - \frac{wv^n}{g} \right)$, find H when $T = 400$, $w = 0.7$, $g = 32.2$, and $v = 80$.

14. Two automobiles are racing on a circular track. One makes the circuit in 31 minutes and the other in $38\frac{1}{2}$ minutes. In what time will the faster machine gain one lap on the slower?

15. The cost of publication of each copy of a certain magazine is $6\frac{1}{2}$ cents. It sells to dealers for 6 cents and the amount received for advertising is 10% of the amount received for all magazines issued beyond 10,000. Find the least number of magazines which can be issued without loss.

16. A tax of 64 cents a pound is laid on woolen cloth. A certain cloth which weighs m ounces per yard and cost \$4 per yard is to be reduced in weight so that it may be sold without decrease of profit, at the same price per yard as before the tax was laid. By how much must the weight of a yard of the cloth be reduced?

17. Simplify $10^0 + 4\frac{1}{2} - 2^{-1} + \frac{1}{\sqrt[3]{-64}} + 8^{\frac{2}{3}}$.

18. Divide $3 - 4x^{-1} + x^{-2}$ by $x^{-1} - 3$.

19. Given $a^2(x^2 - yz)^{-\frac{2}{3}}$. Introduce a^2 into the parenthesis without changing the value of the expression.

20. Simplify $\sqrt{12\sqrt{-6} - 6}$.
21. Given $x^2 - 1.6x + 0.3 = 0$. Compute the value of the larger root correct to three significant figures.
22. Solve $3x^2 - 1.2x - 4.16 = 0$. Obtain result correct to two significant figures.
23. Solve $11x^{\frac{1}{3}} - 6x^{\frac{2}{3}} = 4$.
24. Solve the equation $2x^{\frac{2}{n}} + 3x^{\frac{4}{n}} - 56 = 0$.
25. Solve $(\frac{2}{3})^{4x-7} = (\frac{8}{27})^{2x-5}$.

CHAPTER V

VARIABLES AND LIMITS

THE material of this chapter forms a basis for much of the work of higher mathematics. Without the notion of variables and limits the calculus and all that depends upon it would be impossible.

54. A Variable is a quantity whose value may be considered as changing throughout the problem. A *constant* is a quantity whose value remains fixed. For example, if you take a train from New York to Philadelphia traveling at a uniform rate of 40 miles an hour, the distance you have traveled is an increasing variable, while the distance to Philadelphia is a constant. In the same illustration the time you have been traveling is also an increasing variable. Here the distance traveled depends on the time. When one variable depends on another variable for its value the first is a *function* of the second.

Thus, if x is a variable, the expression $x^2 - 1$ is a variable, and a function of x , because the value of $x^2 - 1$ will depend upon the value of x . If x is a constant, $x^2 - 1$ is also constant.

55. Limits. — When a variable approaches a constant in such a manner that though getting always nearer and nearer the constant it never reaches it, the variable is said to approach the constant *as a limit*. The constant is then said to be the *limit* of the variable.

Consider the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

If we consider successively one term, the sum of two terms, the sum of three terms, etc., this sum is a variable which approaches 1 as a limit. To show this we write the following facts.

- (a) The sum remains always less than 1.
- (b) The sum continually approaches 1.
- (c) There is no number less than 1 which the sum cannot exceed if enough terms are taken.

In a similar manner we can show that the decimal $.333 \dots$ approaches $\frac{1}{3}$ as a limit if the number of places to which it is carried is indefinitely increased.

You should note that a variable may approach a constant when it does not approach it *as a limit*. For example, the distance traveled on the moving train in the illustration of the preceding section was approaching the distance from New York to Philadelphia, and would ultimately equal that distance. Hence the variable in that case was not approaching the constant *as a limit*.

The conditions that a variable shall approach a constant as a limit are

1. That the variable shall constantly approach the constant.
2. That the variable shall never equal the constant.
3. That the variable shall come nearer the constant than any value that can be named.

A variable may approach its limit from either side, that is, it may be always less than its limit, or always greater than its limit, or alternately greater and less.

56. An Infinite Number. — A variable which increases without limit becomes *infinite*.

For example, there is no limit to the sum of the series

$$1 + 2 + 3 + 4 + 5 + \dots,$$

for the sum can be made larger than any number whatever by taking a sufficient number of terms. The sum of such a series is therefore called *infinity* and is denoted by the symbol ∞ . You should notice that the symbol ∞ does not denote any *definite* number, but a number which becomes great *indefinitely*.

57. Symbols. — We use the arrow ($\longrightarrow$) to denote approaching as a limit. Thus $x \longrightarrow k$ is read “ x approaches k as a limit.”

The expression $\lim_{x \rightarrow 3} (x^2 - 1) = 8$ is read, “The limit of $x^2 - 1$, as x approaches 3, is 8.”

58. Meaning of $\frac{k}{0}$. — The expression $\frac{k}{0}$ should not be understood as an actual division of a constant k by absolute 0, since such a division is impossible. As used in this connection $\frac{k}{0}$ means $\lim_{x \rightarrow 0} \frac{k}{x}$.

Since there is no limit to the value of $\frac{k}{x}$ when x is made to approach 0, we write for brevity $\frac{k}{0} = \infty$.

59. Meaning of $\frac{k}{\infty}$. — The symbol $\frac{k}{\infty}$ stands for $\lim_{x \rightarrow \infty} \frac{k}{x}$. As x becomes constantly greater, the value of the fraction $\frac{k}{x}$ becomes constantly smaller. If x is made to increase without limit, the value of the fraction $\frac{k}{x}$ approaches 0 as a limit.

Hence briefly, $\frac{k}{\infty} = 0$.

60. Indeterminate Forms. — The expression $\frac{0}{0}$ has no definite value. The 0's are here used not in the absolute sense, but in the sense of quantities approaching 0 as a limit. Hence the expression $\frac{0}{0}$ may have any finite value whatever. The value of the expression $\frac{0}{0}$ is therefore said to be *indeterminate*. Its value may, however, sometimes be determined in any given case providing we know the variables whose limits the 0's are.

Other indeterminate forms are $\frac{\infty}{\infty}$ and $0 \cdot \infty$.

61. Limit of a Function. — The following examples illustrate the manner in which the limit of a function of x can be found when x approaches a given quantity.

Illustrative example I:

Find $\lim_{x \rightarrow 2} \frac{x^2 - 2}{x^3 + 3}$.

Solution: Substitute $x = 2$ in the function.

Then, $\lim_{x \rightarrow 2} \frac{x^2 - 2}{x^3 + 3} = \frac{4 - 2}{8 + 3} = \frac{2}{11}$.

Illustrative example II:

Find $\lim_{x \rightarrow \infty} \frac{x + 3}{2x^2 - 6}$.

Solution: When x is made sufficiently great the highest power of x transcends all lower powers of x so that their value is negligible in comparison with it. Hence in this case we consider only the ratio of the x^2 terms. The limit is therefore the limit of

$$\frac{0x^2}{2x^2} = \frac{0}{2} = 0.$$

Illustrative example III :

$$\text{Find } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^3 - 8}.$$

Solution : If we substitute $x = 2$ as in the first example above, we obtain $\frac{0}{0}$, which is indeterminate.

$$\text{However, } \frac{x^2 - 4}{x^3 - 8} = \frac{(x + 2)(\cancel{x - 2})}{(\cancel{x - 2})(x^2 + 2x + 4)} = \frac{x + 2}{x^2 + 2x + 4}.$$

If now we substitute $x = 2$, we obtain $\frac{4}{12} = \frac{1}{3}$, which is the limit desired.

Exercises

Find the limit, as x approaches 0, of

$$1. \ x^2 - 3. \quad 2. \ \frac{x + 6}{3x - 2}. \quad 3. \ \frac{5x^2 - 3}{3x + 5}.$$

Find the limit, as x approaches 1, of

$$4. \ x^3 + x^2 - 1. \quad 5. \ \frac{x^2 - 1}{x^2 + 3}. \quad 6. \ \frac{x^2 - 1}{x^2 - 3x + 2}.$$

Find the limit, as x approaches -3 , of

$$7. \ (x + 3)(x - 6). \quad 8. \ \frac{2x^2 + x - 1}{x^3 + 2}. \quad 9. \ \frac{x + 3}{x^2 + 8x + 15}.$$

Find the limit, as x approaches ∞ , of

$$10. \ \frac{x^2 - 3}{2x^2 + x - 1}. \quad 11. \ \frac{3x^3 + 6x^2 + 5}{2x^2 - 8}. \quad 12. \ \frac{x^2 - x + 1}{x^3 - 3}.$$

$$13. \ \frac{x^2 - 9}{x + 3}. \quad 14. \ \frac{x^2 - 3x + 2}{2x^2 - 3x + 4}. \quad 15. \ \frac{x + 1}{x^2 - 1}.$$

Find the limiting values of the following fractions :

$$16. \ \frac{x^2 - 3x + 2}{x^2 + x - 6} \text{ when } x \rightarrow 2.$$

$$17. \ \frac{x^3 + 2x^2 - x - 2}{x^2 + x - 2} \text{ when } x \rightarrow 1.$$

18. $\frac{a^{2x} - 1}{a^x - 1}$ when $x \rightarrow 0$.
19. $\frac{2x + 1}{(4x^2 - 1)^2}$ when $x \rightarrow -\frac{1}{2}$.

Cumulative Review

- Find the limits of x in
 $(x + 1)(x + 2)(x - 3) > (x - 1)(x - 4)(x + 5)$.
- What value of x will make the arithmetic mean between $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ equal to 6?
- Extract the square root of the binomial surd
 $17 + 12\sqrt{2}$.
- Simplify $\frac{3 - 2i}{1 + i}$, and represent the resulting complex number graphically.

5. Put the following expressions in the form $a + bi$, add them, and express the result graphically:

$$(2 + 3i)^2, \frac{3 + \sqrt{-4}}{1 - \sqrt{-4}}, \sqrt{8 - 6i}.$$

6. (a) Reduce $\frac{2 + 3i}{3 - 4i}$ and $\frac{3 + 2i}{3 + 4i}$ to complex numbers.

$$(1 = \sqrt{-1}).$$

(b) Construct graphically the sum and difference of $3 - 4i$ and $-5 + 6i$.

- (c) Express $\left(\frac{1 + i}{2}\right)^3$ and $\sqrt{7 + 24i}$ as complex number.

7. The sum of the first seven terms of an arithmetic progression is 98, and the product of the first and seventh terms is 115. Find the common difference.

8. How many terms of the series 4, 12, 36, $\dots$, must be taken to make the sum 1456?

9. Find the roots of

$$(a) \ x^2 + \frac{1}{x^2} = a^2 + \frac{1}{a^2}. \quad (c) \ \sqrt{3x} + \sqrt{x-2} = \frac{4}{\sqrt{x-2}}.$$

$$(b) \ x^2 + 3x + 3 = 0.$$

10. The distance, S , that a body falls in t seconds from rest is given by the formula $S = 16 t^2$. A man drops a stone into a well and hears the splash after 3 seconds. If the velocity of sound in air is 1086 feet a second, find the depth of the well.

11. (a) Represent $5 + 2i$ and $-3 + 4i$ graphically and prove geometrically that their sum is $2 + 6i$.

$$(b) \text{ Find the value of } \frac{1-2i}{3+i} \cdot \frac{2-i}{1+3i}.$$

12. The sum of three numbers in arithmetic progression is 6. If 1, 2, and 5 respectively are added to the numbers, the three resulting numbers are in geometric progression. Find the numbers.

13. If the square of y varies as the cube of z , and z varies inversely as x , show that xy varies inversely as the square root of x .

$$14. \text{ Simplify } \frac{4i\sqrt{10}}{2-3i\sqrt{5}}, \text{ where } i = \sqrt{-1}.$$

15. Ten potatoes are placed in line at distances 5, 10, 15, $\dots$, feet from a basket. A boy, starting from the basket, picks up the potatoes and carries them back, one at a time, to the basket. How far must he run to complete the potato race?

16. A line 16 inches long is divided into two parts such that the longer part is a mean proportional between the whole line and the shorter part. Find the length of the shorter part to two places of decimals.

17. The perimeter of a certain triangle is 19 feet. The length of one side, which is 6 feet, is a mean proportional between the lengths of the other two sides. Find the length of each side.

18. Given $\frac{\sqrt{4x^2 - 1}}{\sqrt{4x^2 + 1} - \sqrt{4x^2 - 1}} = \frac{\sqrt{-5}}{\sqrt{13} - \sqrt{-5}}$, find x by principles of proportion.

19. Kepler's third law of planetary motion states that the square of a planet's time of revolution is proportional to the cube of its mean distance from the sun. The mean distances of the earth and Mercury from the sun are 93 and 36 millions of miles respectively. Find the time of Mercury's revolution.

20. The simple interest on a fixed sum of money varies as the length of the time it is invested. If a certain sum draws \$145 in 3 months, how much will it draw in 11 months?

21. The numbers a, b, c, d, e, f, g are in arithmetic progression. If $a = -2$ and $g = 12$, find the value of c .

22. On Jan. 1, 1925, a man has \$500. His annual income is \$1200, and his living expenses for the year 1925 will be \$800. If his income remains the same, but the cost of living increases \$20 per year, in how many years will he have \$3600?

23. A man arranges to pay a debt of \$3600 in 40 monthly payments which form an arithmetic progression. After paying 30 of them, he still owes one third of his debt. What was his first payment?

24. Find the limits of x :

$$3x + 1 > 2x + 7.$$

$$2x - 1 < x + 6.$$

25. Solve

$$x - y > 5.$$

$$x + y = 12.$$

CHAPTER VI

PERMUTATIONS, COMBINATIONS, AND PROBABILITY

THE principles of the following chapter are applicable to a large number of interesting problems touching life and experience which cannot be solved by any other process you have had. The topic is taught for its own sake rather than for the sake of its applications to subsequent work. However, there are many applications such as actuarial computations of life insurance problems.

The theorems of this chapter are based on the following

62. Fundamental Principle. — *If an act can be performed in any one of m ways, and when it has been done in any one of those ways a second act can be performed in any one of n ways, then the number of ways of performing the two acts in succession is mn .*

This principle can be extended indefinitely to include any number of acts. The meaning of this theorem can be made clear by considering the following

Illustrative example I:

If there are 6 railroads connecting Chicago and Buffalo, and 5 railroads connecting Buffalo and New York, in how many different ways may a person go by rail from Chicago to New York by way of Buffalo?

Solution: The journey from Chicago to Buffalo can be made in any one of 6 ways: when it has been made in any one of these ways the remaining journey can be made in any one of 5 ways. Hence the number of ways of performing the entire journey is $6 \cdot 5$ or 30 ways.

Illustrative example II :

In a club of 12 boys and 8 girls, in how many ways can a boy and a girl be chosen as president and vice-president respectively of the club?

Solution: The president may be chosen from the boys in 12 ways. The vice-president may be chosen from the girls in 8 ways. Since any girl may be associated with any boy the number of selections is $12 \cdot 8$ or 96 ways.

Illustrative example III :

From 2 sopranos, 3 contraltos, 2 tenors and 4 basses how many quartettes can be formed?

Solution: The soprano singer may be chosen in 2 ways. After she has been chosen, the contralto may be chosen in 3 ways, and any one of the 3 contraltos can be associated with either of the sopranos. Hence the number of ways of choosing a soprano and contralto is $2 \cdot 3$. Continuing in like manner, we find the quartette can be chosen in $2 \cdot 3 \cdot 2 \cdot 4$ or 48 ways.

Exercises

1. A man has 3 coats, 2 vests, and 4 pair of trousers. In how many different costumes may he appear?

2. A boy has 4 pair of shoes. In how many ways may he select a right and left shoe? How many of these selections will not be mates?

3. In a town there are 5 grammar schools. In how many different ways may 3 children attend these schools if no two are in the same school?

4. A house has 16 windows. In how many different ways can a burglar enter by one window and leave by another?

5. A village has 3 meat markets and 4 drug stores. If a boy is sent to buy a piece of meat and a bottle of medicine, in how many ways may these purchases be made?

6. In a school there are 3 science teachers, 4 language teachers, 3 mathematics teachers, and 2 history teachers. If a pupil takes these 4 subjects, in how many ways can his teachers be chosen?

7. From 3 pianists, 4 violinists, and 4 cornetists, how many trios of different instruments can be formed?

8. From a party of 5 boys and 6 girls, how many different couples can be formed for dancing?

9. In how many ways is it possible to post 4 letters in three letter boxes?

10. At a masquerade there are 10 men and their wives. How many different couples can be formed for dancing, if no man dances with his wife?

11. Three coins are pitched. In how many ways can they come up?

12. If two dice are thrown, in how many ways may they fall?

13. How many passenger tickets will a railway company need for use on a division on which there are 20 stations, if only one-way tickets are sold between all stations?

14. Three travelers are in a town where there are only 4 hotels. In how many ways may they put up each at a different hotel?

15. A man has a choice of 5 ways to go to a certain town and a choice of 3 ways by which he must return. In how many ways may he go and return?

16. There are 6 different routes between two cities. In how many different ways may a man travel from one of the cities to the other and return by a different route?

63. A Permutation is an *arrangement* of a number of things in a definite order. Thus, the three letters a , b , and c can be arranged in 6 possible orders, namely abc , acb , bac , bca , cab , and cba , and each of these arrangements is a different permutation of these letters.

Suppose that we wish to know in how many orders Mary, Alice, Henry, and John may be seated on a bench. Let us use their initials to represent them. Then the first seat may be occupied by any one of the four, hence it can be filled in 4 ways. If Mary occupies the first seat, the second seat may be filled by Alice, Henry, or John, that is, in 3 ways. Similarly, if any of the others occupy the first seat the second seat may be filled in 3 ways. Then the third seat can be filled in 2 ways for each way of filling the first two seats. After the first 3 seats are filled, the fourth seat can be filled in only 1 way in each case.

The following diagram shows all possible arrangements, giving M. A. H. J., M. A. J. H., etc., 24 in all. That is, $4 \cdot 3 \cdot 2 \cdot 1$ or 24.

FIRST SEAT	SECOND SEAT	THIRD SEAT	FOURTH SEAT		FIRST SEAT	SECOND SEAT	THIRD SEAT	FOURTH SEAT
M	A . . .	H	J	y	H	M . . .	A	J
		J	H				J	A
	H . . .	A	J			A . . .	M	J
		J	A				J	M
	J . . .	A	H			J . . .	A	M
		H	A				M	A
A	M . . .	H	J	J	J	M . . .	A	H
		J	H				H	A
	H . . .	M	J			A . . .	M	H
		J	M				H	M
	J . . .	M	H			H . . .	M	A
		H	M				A	M

64. Theorem.—The number of permutations of n things taken r at a time is $n(n - 1)(n - 2) \cdots (n - r + 1)$.

Proof: In this case r things are to be chosen from n things and these r things are arranged in every possible order. This is equivalent to filling r different places from n things.

The first place can be filled in n ways. Since one of the n things has now been used, the second place can be filled in $(n - 1)$ ways. In like manner, the third place can be filled in $(n - 2)$ ways, and so on.

When $(r - 1)$ places have been filled, there are $n - (r - 1)$, that is, $n - r + 1$, things left not used. Hence the last place can be filled in $n - r + 1$ ways.

The expression “the number of permutations of n things taken r at a time” may be symbolized thus, ${}_nP_r$.

Then, ${}_nP_r = n(n - 1)(n - 2) \cdots (n - r + 1)$.

In the above discussion it is understood that repetitions are not allowed, that is, that the same thing cannot be used to fill two places. It is also understood that the n things are all different. Note also that the product contains r factors.

Obviously r cannot exceed n . If $r = n$, then,

$${}_nP_n = n(n - 1)(n - 2) \cdots 2 \cdot 1.$$

The product of all numbers from n to 1 inclusive is called *factorial n* and is denoted by the symbol $n!$. For example, $5!$ means $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. Hence,

The number of permutations of n things taken all at a time is factorial n .

Exercises

1. In how many ways can the letters x, y, z be arranged? Write each of these arrangements.

2. (a) In how many ways may 6 boys be seated on a

bench? (b) In how many ways may a bench seating only 4 boys be filled from the 6 boys above mentioned?

3. Four people enter a room in which there are 6 empty chairs. In how many ways may they take their seats?

4. How many numbers of two figures each can be written from the digits 1, 2, 3, and 4? Write the numbers.

5. In how many ways may the letters of the word *Latin* be arranged? How many of these arrangements begin with *L*?

6. In how many ways may a class of 7 boys in spelling be arranged in line, if the same one always stays at the head?

7. How many words of 5 letters each can be made from the letters of the word *capitol*?

8. In how many ways may a guard of 6 men chosen from a detachment of 30 soldiers be arranged in a line?

9. In how many ways may a baseball team be arranged if the same man always pitches, but the others can play in any other position?

10. In how many ways may a number of 5 figures be written from the digits 1 to 9 inclusive without repeating any digit?

11. In how many ways may a number of 5 figures be written from the digits 0 to 8 inclusive without repeating any digit?

12. How many words can be made, using all the letters of the word *decagon*?

NOTE. Any arrangement of letters is considered a word.

13. How many words of 4 letters each can be formed from the letters of the word *New York*?

14. How many arrangements beginning with h and ending with n are possible by using all the letters of the word *heptagon*?

15. How many permutations can be formed with the letters in the word *introduces*?

16. How many different ways are there of arranging the 26 letters of the alphabet, taking four at a time?

17. How many arrangements of the letters of the word *vowel* are possible

(a) Ending in l and beginning with v ?

(b) With w as the middle letter?

18. How many arrangements of the letters of the word *number* are possible

(a) Beginning and ending with a consonant?

(b) Without changing the positions of the vowels?

19. Find the number of permutations of the letters of the word *Camelot* if the vowels must occupy the even places.

20. How many arrangements are possible of the letters of the word *radium* if consonants and vowels alternate?

21. How many numbers between 2000 and 5000 can be written from the digits 2, 4, 3, 9, and 7?

22. How many numbers are there consisting of the ten different digits?

23. If the digits 1, 2, 3, 4, 5, 6, 7, and 8 are arranged in every possible order, how many of the numbers formed will be greater than 50,000,000?

24. Four persons enter an automobile in which there are six seats. In how many ways can they take their seats?

25. How many numbers can be written, using all the digits 1, 4, 7, 9, 2, 6, if the odd and even digits alternate?

26. How many arrangements of the letters $ABCdefg$ are possible beginning and ending with a capital?

27. How many numbers between 3000 and 4000 can be made with the digits 9, 3, 4, 6?

28. In how many different ways can 5 men be seated in 5 seats?

29. If the number of permutations of n things taken 3 at a time is $\frac{2}{5}$ of the number of permutations of $n - 1$ things taken 4 at a time, find n .

65. Permutations with Repetitions.

Theorem: *The number of permutations of n things taken r at a time when each thing may be used any number of times up to r times in any arrangement is n^r .*

Proof: The first place can be filled in n ways. The second place can also be filled in n ways, since the thing used to fill the first place can be used again. In like manner the third place can be filled in n ways, and so, until all the r places are filled. Hence we have $n \cdot n \cdot n \cdots$ to r factors, or n^r . In this case r may be greater than n .

Illustrative example :

In how many ways may 5 coins be given to 2 boys?

Solution: The first coin can be given away in 2 ways since either boy may receive it. The same is true of the second coin and of each of the others.

Hence the result is 2^5 , or 32.

66. Permutations of Things Not All Different.

Theorem: *The number of permutations of n things taken all at a time when p things are of one kind, q of another kind $\cdots$ and the rest different is $\frac{n!}{p!q!\cdots}$.*

Proof: Let x represent the number of permutations under the given conditions. Now if the p things which are alike are replaced by p things which are different, each permutation under the given condition could be made into $p!$ permutations by permuting these p things. The number of possible permutations then become $x \cdot p!$. In like manner if the q like things are replaced by q unlike things the present number of permutations is multiplied by $q!$ and we then have $x \cdot p!q!$. Thus we continue until all the things are different. But we already know that when the n things are all different the number of possible permutations is $n!$.

Hence,
$$x \cdot p!q! \cdots = n!$$

Therefore,
$$x = \frac{n!}{p!q! \cdots}$$

Illustrative example :

Find the number of distinct arrangements of the letters of the word *parallel*.

Solution: There are 8 letters, 3 of which are l and 2 of which are a . The result therefore is $\frac{8!}{3!2!} = 3360$.

67. Circular Permutations.

Theorem: *The number of ways in which n different things can be arranged in a circle is $(n - 1)!$.*

Proof: Let us suppose the n objects to be arranged around a circle, thus forming one circular permutation. Now if this circle is broken at any point and the line straightened out, a certain linear permutation is formed. In like manner the circular permutation might be broken at some other point and straightened out, thus forming a *different* linear permutation. In the same way we may break the circular permutation at n *different* points and straighten it out into n *different* linear permutations.

Hence we see that the one circular arrangement can be made into n linear arrangements, and the same thing is true of every other circular arrangement with which we might start. It follows that the number of linear permutations of the n things is n times the number of circular permutations of the same n things.

$\therefore$ The number of circular permutations of n different things is $\frac{n!}{n} = (n - 1)!$.

Illustrative example :

Find the number of ways in which 6 people may be seated at a round table.

Solution: $(n - 1)! = 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

Exercises

1. In how many ways may the letters of each of the following words be arranged :

Cincinnati? Tennessee? Genesee? Lackawanna?

2. How many numbers of 4 digits each can be written with the digits 2, 3, 7, 5, and 8 when repetition is allowed?

3. If the expression $a^2x^3y^4$ is written at full length, how many arrangements of these letters are possible?

4. In how many ways may 6 children form a ring?

5. In how many ways may 5 men sit at a round table?

6. A bookseller has 3 copies of a certain algebra, 4 copies of a certain Latin book, and 4 copies of a certain history. In how many ways can these books be arranged on a shelf?

7. In how many ways can 7 men be seated at a round table?

8. In how many ways can a bracelet be made by stringing together 7 pearls of different shades?

9. In how many different orders may 10 people form a ring?

10. In how many orders may 4 men and 4 ladies be seated at a round table so that the men and ladies alternate?

Hint: After the ladies are seated, places are thereby definitely established for the men. Hence we have $3! \cdot 4!$.

11. In how many ways may 9 presents be given to two girls so that one receives 3 and the other 6?

Hint: Three are distributed in the same way, and 6 in the same way.

12. In an automobile accident two were killed, 2 injured and the remaining 3 occupants of the car were uninjured. In how many ways may this occur, in so far as the people in the car are concerned?

13. In how many ways may 10 different coins be inserted in two toy banks?

14. Find the number of arrangements of the letters of the word *Tuscarora* so that the vowels occupy the even places.

15. Six men and six ladies are to be seated at a round table. In how many different ways may they be seated in order that there shall be a man at the right of each lady?

16. In how many different ways may 5 white flags and 3 blue flags be arranged one above another?

17. In how many ways can 5 pennies, 6 nickels, and 4 dimes be given to 15 children so that each may receive a coin?

Review

1. Find the value of n , when ${}_nP_4 = 3{}_nP_3$.

2. The sum of 15 terms of an arithmetic progression is 600. The common difference is 5. Find the first term.

3. Find a third proportional to

$$\frac{x^2 - 6x + 9}{x + 4} \text{ and } \frac{x^2 + x - 12}{x + 2}.$$

68. Combinations. — A *combination* is a definite group of things without any regard to their arrangement among themselves. For example, a committee of 3 chosen from a club of 20 members is a combination. A single different member of the committee makes a different combination. The committee is the same, however, regardless of the order in which they might be seated on a bench. The idea of arrangement is, therefore, not involved in combinations.

The symbol for “the number of combinations of n things taken r at a time” is ${}_nC_r$.

Theorem :

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

Proof: Each combination of r things when formed can be arranged in $r!$ ways, and thus made into $r!$ different permutations.

Hence, ${}_nC_r \cdot r! = {}_nP_r$.

$$\text{Therefore, } {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

Corollary: ${}_nC_n = 1$.

Illustrative example :

How many different committees of 4 each can be chosen from a class of 15 members?

$$\text{Solution: } {}_{15}C_4 = \frac{15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4} = 1365.$$

69. Theorem. — The number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $n - r$ at a time, that is, ${}_nC_r = {}_nC_{n-r}$.

Proof: If from n things a group of r things is chosen, there remains a corresponding group of $n - r$ things not chosen. If, then, a *different* group of r things is chosen, the group of $n - r$ things which remains is a *different* group than remained before, so that for every *different* group of r things chosen there remains a corresponding *different* group of $n - r$ things. Hence, the number of possible groups taken r at a time is the same as the number of possible groups taken $n - r$ at a time.

An algebraic proof of this theorem may be given as follows :

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}. \quad \text{If we multiply both}$$

numerator and denominator of this fraction by $(n-r)!$

$$\text{the formula becomes } {}nC_r = \frac{n!}{r!(n-r)!} \quad (1). \quad \text{By applying}$$

this formula to ${}_nC_{n-r}$ we obtain

$${}_nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]} = \frac{n!}{(n-r)!r!} \quad (2).$$

The right members of (1) and (2) are identical ;

$$\therefore {}nC_r = {}nC_{n-r}.$$

Illustrative example :

Find the number of combinations of 20 things taken 17 at a time.

$$\text{Solution: } {}_{20}C_{17} = {}_{20}C_3 = \frac{20 \cdot 19 \cdot 18}{1 \cdot 2 \cdot 3} = 1140.$$

70. Total Number of Combinations of n Things. — By the total number of combinations of n things is meant the total number of groups which can be formed from n things, taking any number at a time from 1 to n . It is therefore equal to the sum ${}_nC_1 + {}nC_2 + {}nC_3 + \cdots + {}nC_n$.

Theorem. — *The total number of combinations of n things is $2^n - 1$.*

Proof: There are two ways of dealing with each one of the n things, that is, it may be chosen or not chosen. Since the first thing may be treated in 2 ways, the second in 2 ways, and so on, the number of ways of dealing with the n things is $2 \cdot 2 \cdot 2 \cdots$ to n factors or 2^n . But this includes the case where each of the n things is rejected. As we do not have a combination in this case, it must be subtracted from the total number of ways of dealing with the n things.

Hence, ${}_nC_{\text{total}} = 2^n - 1$.

Illustrative example :

There are 10 members in a Sunday School class. In how many ways can a committee be appointed from the class?

Solution: Since the number to compose the committee is not specified, it may contain any number from 1 to 10.

Hence, we have $2^n - 1 = 2^{10} - 1 = 1024 - 1 = 1023$.

Exercises

1. Find the number of combinations of 40 things taken 38 at a time.
2. How many different committees each consisting of 3 boys and 2 girls can be appointed from a class consisting of 12 boys and 8 girls?
3. From a party of 10 women and 12 men in how many ways may 2 men and 2 women be chosen to play "doubles" at tennis?
4. If a man has 8 friends, in how many ways may he invite one or more of them to accompany him on a fishing trip?
5. In how many different ways may a committee of 3 be chosen from a club containing 21 members?
6. Find the value of ${}_{25}C_{20}$. Of ${}_{11}C_5$.

7. Find the value of n , when ${}_nC_5 = {}_{n-2}C_5$.

8. How many basketball teams could be selected from a school of 300 if no attention is given to the way the team shall play after its selection?

9. In a geometry class of 25 students, 20 recite each day. In how many ways can these be selected for one day?

10. How many different committees, each composed of 2 Republicans and 3 Democrats, can be formed from 14 Republicans and 21 Democrats?

11. How many straight lines are determined by 12 points, no three of which are in the same straight line? By n points?

12. A man has in his purse a dollar, a half-dollar, a quarter, a dime, a nickel, and a penny. In how many ways can he draw a sum of money from that purse?

13. In how many ways can a baseball team be selected from 36 candidates, if no attention is given to the positions to be played by each person?

14. In how many ways can a football team be selected from 30 candidates, if there is only one man who can play quarterback?

15. From 5 teachers and 100 boys, how many committees can be selected, each containing 2 teachers and 3 boys?

16. After having shaken hands all around, it was found that 66 handshakes had been exchanged. How many persons were in the party?

17. How many different combinations can be made of 16 things taken 5 at a time?

18. In how many ways can a committee of three be selected from ten persons (a) So as always to include one particular person? (b) So as always to exclude that particular person?

19. From 12 men how many committees of 4 can be formed? How many of these would include one particular man A? How many would include A but exclude B?

20. Given 11 points, no three of them lying on a line. How many triangles are there with these points as vertices?

Review

1. Extract the square root of $-3 + 4\sqrt{-1}$.

2. If the number of permutations of n things taken 4 at a time is equal to twice the number of combinations of $n + 1$ things taken 5 at a time, what is the value of n ?

3. A car slipping backward downhill moves 2 inches in the first second, 6 inches in the second second, 10 inches in the third second, and so on. How far will it move in 20 seconds?

Exercises

In solving the examples of the general exercise which follows, you are advised to observe carefully whether the example involves the idea of arrangement, that is, whether it involves permutations or combinations. If it is a permutation problem, note whether repetition is allowed and whether any of the n things are alike.

In many cases the example cannot be solved by a direct application of *any single* formula. You are advised rather to depend on the principles underlying the formulas and to note any special condition or restriction which might modify them. In arranging things some of which are restricted in their movements, such restricted objects should be placed first. It is often helpful to think of *yourself* as actually choosing these groups or arranging them, as the case may be.

1. In how many ways can 4 blue balls, 3 red balls, and 5 white balls be arranged in a line?

2. How many signals can be made by displaying 6 flags of different colors on a vertical staff?

3. How many signals can be made with the above flags if any number of flags from 1 to 6 may be used at a time?

4. How many numbers greater than 54,000 can be written with the digits 2, 5, 6, 7, and 8?

5. How many different numbers of 5 figures each can be written by use of our ten digits when the digits may be repeated? How many numbers of 5 *different* figures each can be made by use of our ten digits?

6. From the digits 3, 4, 9, 8, and 6, how many numbers less than 10,000 can be written?

7. From 10 teachers and 100 pupils how many committees of 5 can be appointed containing at least 2 teachers? Containing just 2 teachers? Containing not more than 2 teachers?

8. Find the number of distinct arrangements of the letters of the word *Mississippi*.

9. From our alphabet containing 21 consonants and 5 vowels, how many words can be formed containing 4 consonants and 2 vowels?

Hint: This is a permutation problem, but we cannot use a permutation formula directly, because a permutation formula both selects and arranges. In this case the number arranged is not the same as the number selected. You will notice that 4 are selected from one group, and 2 from another group, after which the entire 6 are to be arranged in every possible order. We must therefore first select the groups by use of the proper combination formula and then arrange.

Hence we have ${}_{21}C_4 \cdot {}_5C_2 \cdot 6!$

10. How many words each containing 2 vowels and 2 consonants can be made from 4 vowels and 10 consonants?

11. How many words of 5 letters each can be formed from the letters of the word *heptagon* if each word must contain the letter *h*?

12. An algebra class contains 20 members and a history class contains 25 members. In how many orders can 2 pupils chosen from the algebra class and 3 chosen from the history class be seated on a bench?

13. If a pupil has a choice of any 8 questions out of 10 given on an examination, in how many ways can he choose his 8 questions?

14. If my class contains 20 members, in how many ways can I pass 16 and fail 4?

15. In how many ways can 3 men and 3 women sit at a round table so that men and women alternate?

16. How many arrangements are possible of the letters of the word *Onondaga*? In how many of these arrangements do the *o*'s come together?

17. In how many ways can 12 men stand in line so that 2 brothers stand together?

18. In how many ways can 8 women stand in line so that 2 particular women will not be together?

19. How many signals can be made with 5 lights of different colors which may be displayed singly or any number at a time, in a vertical line?

20. Out of 11 business men, 13 physicians, and 8 lawyers, how many different committees can be formed, each consisting of 3 business men, 4 physicians, and 2 lawyers?

21. Find the number of ways that 4 men and 4 ladies can sit at a round table so that no 2 men shall sit together?

22. From the letters of the word *triangles* how many arrangements of 5 letters can be made? Of these, how many consist of 2 vowels and 3 consonants? Of these latter, how many contain *s*?

23. From a school board of 10 men and 6 women how many different committees can be formed composed of 3 men and 2 women?

24. In how many ways can 3 balls be chosen from 6 balls and distributed in 3 boxes so that 1 ball is in each box?

25. With 5 vowels and 6 consonants, how many different words can be formed consisting of 2 different vowels and 3 different consonants, any arrangement of letters being considered a word?

26. Find the value of n , when ${}_8P_n = 24{}_8C_n$.

27. How many numbers of 7 figures each can be formed by use of the digits 5, 5, 0, 0, 2, 1, and 3?

28. There are 20 candidates for a baseball team. Of these, 3 can pitch but cannot play in any other position except 1 of them who can also catch. There are 3 other candidates who can catch but who cannot play in any other position. In how many ways can the team be chosen?

29. How many planes are determined by 12 points of which no 3 are in a straight line?

30. How many diagonals can be drawn in an octagon? A dodecagon? An n -gon?

31. There are 12 points in a plane no three of which are in the same straight line except five which are in the same straight line. How many straight lines can be formed by joining these points?

32. In a small village 435 telephone communications involving different instruments are possible. How many telephones are in the village?

33. If ${}_{20}C_r = {}_{20}C_{4r}$ find the value of r .
34. If ${}_nP_4 = 2({}_{n+1}C_5)$, find the value of n .
35. In how many ways can a class of 20 be divided into 2 equal parts?
36. How many arrangements are possible, using all the letters of the word *isosceles*
- (a) Ending with *les*?
 - (b) Beginning with *i*?
 - (c) With the vowels in their present positions?
 - (d) So that all the *s*'s occur together?
37. If ${}_{12}C_r = {}_{12}C_{r+4}$, find ${}_rC_2$.
38. In how many ways can 5 boys and 4 girls be arranged in a line so that all the boys stand together and all the girls stand together?
39. In how many ways can the letters of the word *Albany* be arranged if the vowels occupy the odd places?
40. If out of a detachment of 50 soldiers, 5 are chosen each night for guard duty, for how many nights can a different guard be formed?
41. If I have 15 friends, in how many ways can I invite one or more as guests for the week-end?
42. How many selections of three different coins can be chosen from a bag containing 12 different coins?
43. How many different sums of money can be drawn from the above bag?
44. If ${}_nC_3 : {}_{n+2}C_4 = 4 : 15$, find n .
45. 3 ladies and 5 men enter a room in which there are 2 sofas, 4 rockers, and 2 straight chairs. In how many ways can they be seated if one man must occupy a sofa and the ladies must sit in rockers?
46. State the ratio between ${}_{20}P_5$ and ${}_{20}C_5$.

47. Find the value of n , when ${}_nP_2 = 24 {}_nC_{n-1}$.
48. How many different signals can be made with 10 flags, of which 2 are red, 3 white, and the rest blue, always hoisted together and one above another?
49. In how many different ways can 6 balls be distributed in 3 boxes, so that 3 balls are in the first box, 2 in the second, and 1 in the third?
50. A committee of five is to be chosen from a group of seven Englishmen and eight Americans. In how many ways can the committee be chosen if it is to contain just two Englishmen?
51. A coach seats 5 people on each side. Seven people enter the coach, 2 of whom must sit on the right side. In how many ways can they be seated?
52. In how many ways can the crew of an 8-oared boat be arranged, if 2 must row on the port side and 2 others must row on the starboard side?
53. In how many different orders can 3 white balls, 2 blue balls, and 4 red balls be arranged around a circle?
54. In a trial by jury, 3 vote for conviction and 9 for acquittal. In how many ways can this be done?
55. From 10 Englishmen and 12 Americans it was desired to appoint a committee of 6 so that the number of Englishmen on the committee should not exceed the number of Americans. In how many ways can this be done?
56. In how many ways can a committee of 5 be appointed from the U. S. Senate, if the "majority leader" is always a member of the committee?
57. Prove that ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$.

58. If n is an even number, show that ${}_nC_r$ has the greatest value when $r = \frac{n}{2}$.

59. If n is an odd number, show that ${}_nC_r$ is greatest when $r = \frac{n+1}{2}$ or $\frac{n-1}{2}$ and that these values of r give the same result.

60. Find the greatest number of straight lines in which 15 planes can intersect.

71. Probability or Chance. — If in n possible trials, an event may happen in x ways and fail to happen in y ways, where $x + y = n$, and each of these ways is equally liable to occur, then the probability that the event will happen is $\frac{x}{n}$ and the probability that it will fail is $\frac{y}{n}$. That is, the fraction which represents the number of favorable happenings divided by the total number of happenings represents the probability of a favorable outcome.

Example. — A boy has 7 relatives, 3 uncles and 4 aunts. He is to visit one of them, which one to be determined by lot. What is the probability that he will visit an aunt? Here there are seven possibilities. Four of these possibilities are that he visit an aunt. Hence the probability that he does so is $\frac{4}{7}$. Also the probability that he visits an uncle is $\frac{3}{7}$. Hence the visit to an aunt is more probable than a visit to an uncle in the ratio of 4 to 3. This is expressed by saying that the odds in favor of visiting an aunt is 4 to 3, and the odds against it is 3 to 4.

From the above definition it becomes clear that a probability of 1 represents *certainty*. Hence if the probability that

an event will happen is $\frac{x}{n}$, then the probability that it will

not happen is $1 - \frac{x}{n}$. That is, $\frac{y}{n} = 1 - \frac{x}{n}$.

If an event is certain to fail, the probability that it will happen is 0.

Illustrative example I:

A committee of 3 is to be chosen by lot from 8 boys. What is the probability that A, B, and C compose the committee? What is the odds against D being in it?

Solution: The number of possible committees of 3 which can be chosen from 8 boys is ${}_8C_3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$. Hence

the probability that the committee consists of A, B, and C is $\frac{1}{56}$. The number of possible committees which do not

contain D is, ${}_7C_3 = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35$. Hence the number which

do contain him is $56 - 35 = 21$. Hence the odds against D being on the committee is 35 to 21, or 5 to 3.

Illustrative example II:

A purse contains 6 dimes, 4 nickels, and 5 pennies. If 4 coins are drawn at random from this purse, what is the chance that they will be 3 nickels and a dime?

Solution: The total number of ways of choosing 3 nickels out of 4 is ${}_4C_3 = 4$. The total number of ways of choosing 1 dime out of 6 is 6. Hence, the number of ways of making the above choice is $4 \cdot 6 = 24$. The total number of ways of choosing 4 coins out of 15 is

$$\frac{{}^{15}P_4}{{}^4P_4} = \frac{15 \cdot 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4} = 1365.$$

Hence the probability is $\frac{24}{1365} = \frac{8}{455}$.

Illustrative example III :

A lottery offers first and second prizes. If 20 tickets are sold, of which A owns 4, what is the probability that he draws at least one prize?

Solution: A may draw 2 prizes and 2 blanks in ${}_2C_2 \cdot {}_{18}C_2 = 1 \cdot 153 = 153$ ways. He may draw 1 prize and 3 blanks in ${}_2C_1 \cdot {}_{18}C_3 = 2 \cdot 816 = 1632$ ways. Hence the total number of ways of drawing a prize is the sum of 153 and 1632, or 1785. But the total number of ways of drawing 4 tickets out of 20 is ${}_{20}C_4 = 4845$. Hence A's chance of success is $\frac{1785}{4845} = \frac{119}{323}$.

72. Compound Probability. — The probability that several unrelated events will occur in succession is the product of their respective probabilities.

Illustrative example :

A coin is tossed 3 times in succession. What is the probability that it comes down heads up each time?

Solution: Since the coin may fall in either of two ways and they are equally probable, the chance that it will fall heads up on the first throw is $\frac{1}{2}$. There is the same chance that it will fall heads up on each succeeding throw. Hence, the chance that it will fall heads up on all three throws is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

Exercises

1. If your name is one of 250 in a jury wheel, and 12 jurymen are to be drawn, what are the odds against your being called to serve?
2. If there are 2 brothers in a class of 8 boys and these boys are lined up at random, what is the chance that the brothers will be together?

3. Four men and 4 women are seated at random at a round table. What is the probability that men and women alternate?

4. If 3 balls are drawn from a bag containing 3 red balls, 2 white balls, and 1 black ball, what is the chance that the balls drawn are all red?

5. A class of 12 pupils contains only 2 girls. If 6 chosen by lot are to be seated at random on a bench, what is the probability that the girls will be seated at the ends of the bench?

6. The odds are 5 to 3 that one football team A will defeat another football team B. What does this mean? How could this be shown by drawing balls from a bag which contained 8 balls, 5 of which are white?

7. If the odds are 10 to 1 against an event, what is the probability of its happening?

8. The chance of an event is $\frac{2}{9}$. Find the odds for or against the event.

9. What is the chance of a year, not a leap year, having 53 Sundays?

10. If 4 cards are drawn from a pack of 52 cards containing 13 hearts, what is the chance that they will all be hearts?

11. If two letters are selected at random out of the alphabet, what is the chance that both will be vowels? (*y* and *w* are considered consonants.)

12. A person is allowed to draw 2 tickets from a bag containing 40 blank tickets and 10 tickets each entitling the holder to a prize of \$100. What chance has he of drawing a prize?

13. If three cards are drawn from a pack, what is the chance that they will be king, queen, and jack of the same suit?

14. In playing a game of dominoes which go no higher than double sixes, a player draws a hand of 5 dominoes. What is the chance that they are all doubles?

15. If two dice are thrown, what is the chance that both will turn up 3?

16. A lottery sells 20 tickets and offers two prizes, an automobile and a piano. If A buys 2 tickets, what is the chance that he gets the automobile? The piano? Both? Neither?

17. If a die is thrown 5 times in succession, what is the chance that a 6 is not thrown?

18. A child playing with blocks, places the letters i, s, m, c, u together at random. What is the chance that they form the word *music*?

19. In example 18, what are the odds in favor of the vowels being together?

20. From a group of 6 women and 4 men, a committee of 5 is chosen by lot. What is the chance that there will be more men than women on the committee?

21. 12 persons made a journey by rail and two of the number were killed. The survivors returned by boat and 3 were drowned. If your friend was one of the party, what is the probability (a) that he was killed? (b) that he was drowned? (c) that he survived?

22. If 2 dice are thrown, what is the chance that both are 1's? That one is a 1 and the other a 2?

23. If the odds in favor of an event's occurring is 3 to 7 and trial is made 1000 times, how many times do you expect it to occur?

24. In a bus wreck there were reported 5 injured out of the 20 passengers. One family of 7 was in the bus. What was the chance that all of the injured belonged to that family?

25. A father forty-seven years of age has a son thirteen years of age. What is the probability that both will be alive 25 years hence?

By the use of the table of mortality given on page 114 which is the Actuaries' Table of Mortality in use by English life insurance companies and many American companies, we note that of 72,582 persons who are forty-seven 31,159 will probably live to be seventy-two years of age; that of 97,978 persons of thirteen years of age, 80,253 will probably live to be thirty-eight years of age. Therefore, the number of cases favorable to both is $31,159 \times 80,253$. Similarly the whole number of cases is $72,582 \times 97,978$.

The probability of the above problem then is $\frac{31,159 \times 80,253}{72,582 \times 97,978}$. The value to four decimals may be

found either by logarithms or by simple arithmetic to be .3516+ or, to put it another way, the chances are 3516 out of 10,000.

Referring to the table on page 114, find the probability in each of the following:

26. That a man of 40 will live to be 55. To be 65. To be 70. To be 85. That he will die within 10 years. Within 15 years. Within 25 years.

27. That a man of 80 will live 2 years. 5 years. 8 years.

28. That a boy of 12 will live to be 40. To be 50. To be 60. To be 70. To be 80. To be 90.

AGE	NUMBER LIVING	NUMBER DYING	AGE	NUMBER LIVING	NUMBER DYING	AGE	NUMBER LIVING	NUMBER DYING
10	100,000	676	40	78,653	815	70	35,837	2,327
11	99,324	674	41	77,838	826	71	33,510	2,351
12	98,650	672	42	77,012	839	72	31,159	2,362
13	97,978	671	43	76,173	857	73	28,797	2,358
14	97,307	671	44	75,316	881	74	26,439	2,339
15	96,636	671	45	74,435	909	75	24,100	2,303
16	95,965	672	46	73,526	944	76	21,797	2,249
17	95,293	673	47	72,582	981	77	19,548	2,179
18	94,620	675	48	71,601	1,021	78	17,369	2,092
19	93,945	677	49	70,580	1,063	79	15,277	1,987
20	93,268	680	50	69,517	1,108	80	13,290	1,866
21	92,588	683	51	68,409	1,156	81	11,424	1,730
22	91,905	686	52	67,253	1,207	82	9,694	1,582
23	91,219	690	53	66,046	1,261	83	8,112	1,427
24	90,529	694	54	64,785	1,316	84	6,685	1,268
25	89,835	698	55	63,469	1,375	85	5,417	1,111
26	89,137	703	56	62,094	1,436	86	4,306	958
27	88,434	708	57	60,658	1,497	87	3,348	811
28	87,726	714	58	59,161	1,561	88	2,537	673
29	87,012	720	59	57,600	1,627	89	1,864	545
30	86,292	727	60	55,973	1,698	90	1,319	427
31	85,565	734	61	54,275	1,770	91	892	322
32	84,831	742	62	52,505	1,844	92	570	231
33	84,089	750	63	50,661	1,917	93	339	155
34	83,339	758	64	48,744	1,990	94	184	95
35	82,581	767	65	46,754	2,061	95	89	52
36	81,814	776	66	44,693	2,128	96	37	24
37	81,038	785	67	42,565	2,191	97	13	9
38	80,253	795	68	40,374	2,246	98	4	3
39	79,458	805	69	38,128	2,291	99	1	1

29. A man 20 years of age marries a lady 18 years of age. What is the probability that they will live to celebrate their silver wedding anniversary? Their golden wedding? Their diamond wedding?

30. A man 22 years of age marries a lady 20 years of age. What is the probability that they both will live to see their son married 25 years from the date of their wedding?

Cumulative Review

1. Simplify $\frac{a^2 - (b + c + a)^2}{(a + b)^2 - (a + c)^2}$.

2. Write as the difference of two squares:

$$(2x^2 - 3y^2 + 4z^2)(2x^2 - 3y^2 - 4z^2).$$

3. Construct the points $2 + 2i$ and $2 + i$ and show that their sum $4 + 3i$ corresponds to the fourth vertex of a parallelogram of which the origin and the two given points are the other three.

4. Express algebraically

(a) Nine more than the fourth power of a number is equal to ten times the square of the number.

(b) The number which exceeds a by the sum of the squares of b and c .

(c) The number which if 5 be subtracted from it, and 1 be added to the square of the remainder, the sum will be 10.

(d) A number of three digits, the first, second, and third being x , y , and z respectively.

5. A certain number was to be added to $\frac{1}{2}$, but by mistake $\frac{1}{2}$ was divided by the number. However, the correct result was obtained. What was the number?

6. From a thread, whose length is equal to the perimeter of a square, 36 inches are cut off, and the remainder is equal in length to the perimeter of another square whose area is four ninths that of the first. What is the length of the thread?

7. The floor of a room contains $30\frac{1}{3}$ square yards; one wall contains 21 square yards, and an adjacent wall contains 13 square yards. What are the dimensions of the room?

8. Solve the following proportion for the value of x .

$$a + b + \frac{2b^2}{a - b} : \frac{(a + b)^2}{2ab} - 1 = x : a - b.$$

9. The ratio of a father's age to his son's age is 9 : 5. If the father is 28 years older than the son, how old is each?

10. A horse is tied by a rope 20 yards long in the center of a field and eats all the grass within its reach in $2\frac{1}{2}$ days. How many days would it have taken the horse to eat all the grass within its reach if the rope had been 10 yards longer? Area of the circle varies as the square of the radius.

11. The sum of the terms of an arithmetic progression of six terms is 66, and the sum of the squares of the terms is 1006. Find the elements of this progression.

12. The sum of the first four terms of a geometric progression is 15, and the sum of the terms from the second to the fifth inclusive is 30. What is the first term and the ratio?

13. The arithmetic mean between two numbers is 10, and the geometric mean is 6. What are the numbers?

14. Find the limiting value of the following fractions:

(a) $\frac{x^2 - 6x + 5}{x^2 - 8x + 15}$ when $x \longrightarrow 5$.

(b) $\frac{x^3 - 3x + 2}{x^2 - 7x + 5}$ when $x \rightarrow 1$.

15. If $2x + \frac{1}{2}x - 4 > 6$, prove that $x > 4$.

16. If $3x - 7 - 5x > \frac{3}{2}x - 9$, what limiting value does x have?

17. The perimeters of two similar polygons are expressed by $2x^2 + 3x$ and $3x + 5$ and two corresponding sides are expressed by .03 and .05. Find values of x , and then find the perimeters.

18. Subtract the following graphically, locating the minuend, subtrahend, and remainder points by the use of the parallelogram method.

(a) $(10 + i) - (2 - 6i)$.

(b) $(-4 + 9i) - (-6 + 4i)$.

19. Show graphically that the sum of two conjugate complex numbers is a real number.

20. Solve $v = 332.4\sqrt{1 + 0.003665t}$ for the value of t .

21. Simplify

$$\frac{\frac{a(b-c)}{b+c} + \frac{b(c-a)}{c+a} + \frac{c(a-b)}{a+b}}{\frac{a(b-c)}{(a+b)(a+c)} + \frac{b(c-a)}{(b+c)(b+a)} + \frac{c(a-b)}{(c+a)(c+b)}}$$

22. Solve $\frac{3x - \frac{3x-1}{x+1}}{x + \frac{x-1}{x+1}} = 3$.

23. To correct a barometer reading for temperature, the following amount is subtracted from the reading:

$$B \cdot \frac{m(t-32) - s(t-62)}{1 + m(t-32)},$$

where B is the barometer reading in inches, t the temperature in degrees Fahrenheit, $m = 0.00010$, $s = 0.00001$. What is the corrected reading of the barometer when the temperature is 75° and the barometer reads 29.25 inches?

24. Simplify and express with positive indices:

$$(a) \left[\frac{(a^{-\frac{2}{3}}x^{\frac{1}{6}})^3}{(x^{-\frac{1}{2}}a^{\frac{3}{8}})^{-\frac{1}{2}}} \right]^{\frac{1}{2}} \quad (b) \sqrt{a^{\frac{9}{2}}\sqrt{a^{-3}}} \div \sqrt{(\sqrt{a^{-7}})a^{\frac{1}{3}}}.$$

25. A student was to find the area of a circle. By mistake he used the value 3.41 instead of 3.14, the value of π ; his answer was consequently 10.2 too large. Find the radius of the circle.

CHAPTER VII

THE BINOMIAL THEOREM

IN your study of elementary algebra you have learned how to square or cube any binomial mentally. It is frequently necessary, however, to raise a binomial to a power higher than the third, and it is desirable that we have a method of doing this which shall be shorter than multiplication. The method discussed in this chapter is also important historically and in its application to higher mathematics.

73. The Binomial Theorem. — The binomial theorem is a method by means of which any binomial may be raised to any power. It is stated in formula form as follows:

$$(a + b)^n = a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 \\ + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \dots + \frac{n(n-1) \cdot \dots \cdot 1}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} a^0b^n.$$

In studying this formula you will notice that the coefficients of the respective terms are identical with certain formulas learned in connection with *combinations*.

For example, ${}_nC_1 = \frac{n}{1}$.

$${}_nC_2 = \frac{n(n-1)}{1 \cdot 2}.$$

$${}_nC_3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}.$$

And ${}_nC_n = \frac{n(n-1) \cdot \dots \cdot 1}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} = 1$. Also $a^0 = 1$.

Hence the binomial theorem may be written

$$(a + b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + {}_nC_3 a^{n-3}b^3 + \cdots {}_nC_n b^n.$$

We should recall, however, that such expressions as ${}_nC_2$ are meaningless unless n is a positive integer. The binomial theorem may, however, be proved true for all values of n . This form of expansion, when n is negative or fractional may be proved by a knowledge of calculus and found true except for the last term, when a is numerically larger than b .

Hence, in writing this formula by use of the symbols ${}_nC_1, {}_nC_2, {}_nC_3$, etc., we are giving them an enlarged meaning.

That is, we are defining ${}_nC_2$ as equal to $\frac{n(n-1)}{1 \cdot 2}$, etc., regardless of the value of n .

The further work of memorizing this formula is very easy when we note that it is homogeneous and of the n th degree with reference to a and b , the exponents of a descending from n to 0, and the exponents of b ascending from 0 to n . From this we observe that

The first term is a^n .

The last term is b^n .

The exponents of a decrease by one, beginning with the second term.

The exponents of b increase by one, beginning with the third term.

The coefficient of any term is found by multiplying the coefficient of a in the preceding term by the exponent of a in the preceding term and the result divided by one more than the exponent of b in the preceding term.

The sign of each term of the expanded form is plus if both a and b are plus, and the signs of the even terms are minus if the sign of b is minus, when n is a positive integer.

The number of terms of the expansion is one greater than n , when n is a positive integer.

74. Application of the Binomial Theorem. — When the exponent n is a positive integer, the number of terms in the expansion is $n + 1$.

In expanding by the Binomial Theorem it is always best to substitute first in the formula and then to simplify, rather than to try to combine the two processes.

Illustrative example :

Expand to 4 terms $(x^{\frac{2}{3}} - \frac{1}{2})^{-3}$.

There is an infinite number of terms in the expansion when n is not a positive integer, hence there is no *last* term.

Here $a = x^{\frac{2}{3}}$, $b = (-\frac{1}{2})$, $n = -3$.

Substituting in the Binomial Formula, we have

$$\begin{aligned} \left(x^{\frac{2}{3}} - \frac{1}{2}\right)^{-3} &= (x^{\frac{2}{3}})^{-3} + \frac{-3}{1} (x^{\frac{2}{3}})^{-4} \left(-\frac{1}{2}\right) \\ &\quad + \frac{(-3)(-4)}{1 \cdot 2} (x^{\frac{2}{3}})^{-5} \left(-\frac{1}{2}\right)^2 \\ &\quad + \frac{(-3)(-4)(-5)}{1 \cdot 2 \cdot 3} (x^{\frac{2}{3}})^{-6} \left(-\frac{1}{2}\right)^3 + \dots \end{aligned}$$

Simplifying, we obtain

$$(x^{\frac{2}{3}} - \frac{1}{2})^{-3} = x^{-2} + \frac{3}{2} x^{-\frac{8}{3}} + \frac{3}{2} x^{-\frac{10}{3}} + \frac{5}{4} x^{-4} + \dots,$$

or, if written with positive exponents,

$$= \frac{1}{x^2} + \frac{3}{2x^{\frac{8}{3}}} + \frac{3}{2x^{\frac{10}{3}}} + \frac{5}{4x^4} + \dots$$

75. The R th or General Term of a Binomial Expansion. —

It is often desirable to write a specific term of a binomial expansion without writing the whole expansion. For example, we might wish to obtain the 6th term in the expansion

of $\left(2a^2 - \frac{x}{3}\right)^8$.

By a study of the formula we notice that the coefficient will be ${}_nC_5$, since the series of combination formulas used as coefficients begin in the second term. Also since the powers of b or $\left(-\frac{x}{3}\right)$ begin in the second term and ascend, then in the 6th term we shall have $\left(-\frac{x}{3}\right)^5$. Since the sum of the exponents of a and b is 8, the exponent of a or $2a^2$ must be 3.

Hence the 6th term is

$$\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (2a^2)^3 \left(-\frac{x}{3}\right)^5 = 56 \cdot 8 a^6 \left(-\frac{x^5}{243}\right) = -\frac{448}{233} a^6 x^5.$$

This may be expressed in formula form thus,

$$\text{Rth term} = \frac{n(n-1)(n-2) \cdots (n-r+2)}{1 \cdot 2 \cdot 3 \cdots (r-1)} a^{n-r+1} b^{r-1},$$

where r represents the number of the term to be obtained.

Explanation: The number, one less than the number of the term, may be called the *key number*, since it is the exponent of b and the last factor in the denominator of the coefficient, and, subtracted from n , it gives the exponent of a . The number of factors in the numerator of the coefficient is one less than the number of the term.

Illustrative example I:

Find the 4th term of $(a-x)^{27}$.

Solution: The exponent of x is 1 less than 4, or 3, and the exponent of a is $27-3$, or 24. Since the number of the term indicates minus, we have

$$\text{4th term} = -\frac{9 \cdot 13}{1 \cdot 2 \cdot 3} \frac{27 \cdot 26 \cdot 25}{a^{24} x^3} = -2925 a^{24} x^3.$$

Illustrative example II:

Find the 40th term of $(a+x)^{43}$.

Solution: The exponent of x is 39 and the exponent of a is $43 - 39$, or 4.

$$\therefore 40\text{th term} = \frac{43 \cdot 42 \cdot 41 \cdot 40 \cdot 39 \cdots 5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 39} a^4 x^{39}.$$

If we recall that in the expression $(a + x)^{43}$ there are 44 terms and that the 40th term is therefore the 5th term from the end, we may find the coefficient of the 40th term by finding the coefficient of the 5th term, for their coefficients are ${}_{43}C_4$ and ${}_{43}C_{39}$ respectively, and ${}_nC_r = {}_nC_{n-r}$.

Illustrative example III :

Find the 9th term in the expansion of $\left(x^2 + \frac{1}{x}\right)^{-2}$.

Solution: Here $a = x^2$, $b = \frac{1}{x}$, $n = -2$, $r = 9$.

By the formula, 9th term

$$\begin{aligned} &= \frac{(-2)(-3)(-4)(-5)(-6)(-7)(-8)(-9)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} (x^2)^{-10} \left(\frac{1}{x}\right)^8 \\ &= \frac{9}{x^{28}}. \end{aligned}$$

Illustrative example IV :

Write the term which contains x^5 in the expansion of

$$\left(x^2 + \frac{1}{x}\right)^{10}.$$

Solution: From r th term formula we see that

$$(x^2)^{n-r+1} (x^{-1})^{r-1} = x^5,$$

$$\text{or } (x^2)^{10-r+1} (x^{-1})^{r-1} = x^5,$$

$$\text{or } x^{20-2r+2} x^{1-r} = x^5,$$

$$\text{or } x^{23-3r} = x^5.$$

$$\therefore 23 - 3r = 5,$$

$$\text{and } r = 6.$$

Hence, the desired term is the 6th term.

$$\text{The 6th term} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} (x^2)^5 \left(\frac{1}{x}\right)^5 = 252 x^5.$$

Exercises

Expand completely :

1. $(a - 3b)^3$.

4. $\left(\sqrt{x} - \frac{3}{2x}\right)^6$.

2. $(2x^2 - 1)^5$.

5. $(m^{\frac{1}{3}} + 2m)^5$.

3. $\left(\frac{x}{2} + \frac{2}{x^2}\right)^5$.

6. $(1 + 3x^{\frac{2}{3}})^6$.

7. $(x^{-\frac{1}{2}} + 2y^{\frac{1}{3}})^6$.

Expand to 4 terms :

8. $(1 + x)^{-1}$.

12. $(a^2 - x^{-3})^{\frac{1}{2}}$.

9. $\left(\frac{x^2}{a} - a\right)^{-2}$.

13. $\left(x - \frac{\sqrt{y}}{2}\right)^{-\frac{1}{2}}$.

10. $\left(x^{\frac{2}{3}} + \frac{1}{x}\right)^{-3}$.

14. $\left(\frac{x}{2} + \frac{2}{y}\right)^{-2}$.

11. $(1 - x)^{\frac{1}{2}}$.

15. Write the 8th term in the expansion of $\left(1 - \frac{x^2}{2}\right)^{10}$.

16. Write the 6th term in the expansion of $\left(2 + \frac{a^2}{2}\right)^{20}$.

17. Write the 10th term in the expansion of $(a + \sqrt{b})^{-1}$.

18. Write the 8th term in the expansion of $\left(2 + \frac{1}{2a^{-1}}\right)^{-3}$.

19. Write the middle term in the expansion of $(x^2 - 6y^{\frac{2}{3}})^8$.

20. Write the term which does not contain x in the expansion of $\left(x^3 + \frac{3}{x^2}\right)^{15}$.

21. By use of the Binomial Theorem extract the cube root of 7 to three decimal places.

Hint: $\sqrt[3]{7} = 7^{\frac{1}{3}} = (8 - 1)^{\frac{1}{3}}$. After expanding, change the terms to decimal form and combine them.

22. Extract to four decimal places the fifth root of 30.
23. Extract to four decimal places the fourth root of 18.
24. The amount at compound interest of \$1000 for 5 years at 6% compounded annually may be found by the formula $A = 1000(1.06)^5$. Compute the amount, writing $(1.06)^5$ as $(1 + \frac{3}{50})^5$ and expanding by the Binomial Theorem.
25. Find the 28th term of $(x^2 - y^{\frac{1}{5}})^{31}$.
26. Write the middle term and the term following it in the expansion of $(a^2 - a^{-2})^{20}$. Simplify the exponents, but do not multiply out the coefficients.
27. Find the 5th term of $(\frac{2x^2}{3} - \frac{3}{2x})^{12}$.
28. Find the middle term of $(2a^3 - \frac{3b^2}{4})^8$.
29. Find the middle term of $(3x + \frac{1}{3x})^{20}$.
30. Show that the coefficient of the middle term of $(1+x)^{16}$ is equal to the sum of the coefficients of the 8th and 9th terms of $(1+x)^{15}$.
31. Find the term of $(\frac{4x}{5} - \frac{5}{4x})^{10}$ which is independent of x .
32. Expand $(\frac{2x}{y} - y^2)^5$. Check result by substituting 2 for x , and 1 for y .
33. Write the first three and the last three terms of the expansion of $(2x^2 + \frac{1}{\sqrt{x}})^8$, and simplify the results.
34. Write without negative exponents the expansion of $(a^{-1}bx^{\frac{1}{2}} + a^{\frac{1}{2}}b^{-\frac{1}{2}}x^{-\frac{1}{4}})^4$.

35. Expand $\left(a^{-\frac{1}{6}} + \frac{b^2}{2}\right)^{-4}$ to five terms.

36. Compute $(0.99)^5$ by expanding $(1 - .01)^5$ by the Binomial Theorem and then simplifying.

37. Find correct to three decimal places $(.9)^8$.

38. Find the value of $(1.02)^{-5}$ to the fourth decimal place.

39. Prove that

$$(a + b)^7 - a^7 - b^7 = 7ab(a + b)(a^2 + ab + b^2)^2.$$

40. Find the coefficient of x^{18} in $\left(x^2 + \frac{3a}{x}\right)^{15}$.

41. The amount at compound interest of \$2000 for 4 years at 6% compounded annually may be found by the formula $A = 2000(1.06)^4$. Compute the amount by referring to example 24 above.

42. By the method of example 24, find the amount of \$5000 at compound interest for 7 years at 4% compounded annually.

43. Find the amount of \$10,000, for 6 years with interest compounded annually at 5%.

44. The amount of \$2000 for 4 years at 6% compounded semiannually may be found by the formula $A = p\left(1 + \frac{r}{2}\right)^{2n}$, or $A = 2000(1.03)^8$. Compute it.

76. Proof of Binomial Theorem by Mathematical Induction. — This form of expansion except for the last term is true for every value of n provided a is greater than b , but the proof for the general case is considered beyond the range of this book. When n represents any positive integer, it may be proved by a method which is called *mathematical induction*.

The proof follows.

We know that

$$(a + b)^2 = a^2 + \frac{2}{1} ab + \frac{2 \cdot 1}{1 \cdot 2} b^2,$$

$$\text{and } (a + b)^3 = a^3 + \frac{3}{1} a^2 b + \frac{3 \cdot 2}{1 \cdot 2} ab^2 + \frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} b^3.$$

Hence these two expressions do follow the form stated in the general theorem. Now let n represent any value of the exponent for which the formula is known to be true, as, for example, 3. Then for that value of n ,

$$\begin{aligned} (a + b)^n &= a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \dots \\ &+ \frac{n(n-1)(n-2) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots (r-1)} a^{n-r+1}b^{r-1} \\ &+ \frac{n(n-1) \dots (n-r+1)}{1 \cdot 2 \cdot 3 \dots r} a^{n-r}b^r + \dots. \end{aligned}$$

Now multiply both members of this identity by $(a + b)$.

$$\begin{aligned} (a + b)^n &= a^n + \frac{n}{1} a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \dots \\ &+ \frac{n(n-1) \dots (n-r+2)}{1 \cdot 2 \dots (r-1)} a^{n-r+1}b^{r-1} \\ &+ \frac{n(n-1) \dots (n-r+1)}{1 \cdot 2 \dots r} a^{n-r}b^r + \dots. \end{aligned}$$

$a + b$

$$\begin{aligned} (a + b)^{n+1} &= a^{n+1} + \frac{n}{1} a^n b + \frac{n(n-1)}{1 \cdot 2} a^{n-1}b^2 + \dots \\ &+ \frac{n(n-1) \dots (n-r+1)}{1 \cdot 2 \dots r} a^{n-r+1}b^r + \dots \\ &+ a^n b + \frac{n}{1} a^{n-1}b^2 + \dots \\ &+ \frac{n(n-1) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots (r-1)} a^{n-r+1}b^r + \dots. \end{aligned}$$

$$(a+b)^{n+1} = a^{n+1} + \frac{n+1}{1} a^n b + \frac{(n+1)n}{1 \cdot 2} a^{n-1} b^2 + \dots \\ + \frac{(n+1)(n-1) \cdot \dots \cdot (n-r+2)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot r} a^{n-r+1} b^r + \dots$$

Since this result is of the same form with reference to the exponent $n+1$, that the original formula is with reference to the exponent n , it follows that, if this is a true form for the exponent n , it is also a true form for the exponent $n+1$. In other words, if the theorem is true for *any* exponent, it is true for the *next higher* exponent. But it is true for exponent 3, as has already been shown. Hence it is true for exponent 4. Being true for 4, it is true for 5, and so on indefinitely. Hence, the theorem may be proved true for any positive integral exponent.

Other Theorems Connected with the Binomial Formula

77. Theorem. — *In any binomial expansion the sum of the binomial coefficients is 2^n .*

Proof: In the identity

$$(a+b)^n = a^n + {}_nC_1 a^{n-1} b + {}_nC_2 a^{n-2} b^2 + \dots + {}_nC_n.$$

Let $a = 1$ and $b = 1$, then

$$2^n = 1 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n.$$

78. Theorem. — *The total number of combinations of n things is $2^n - 1$.*

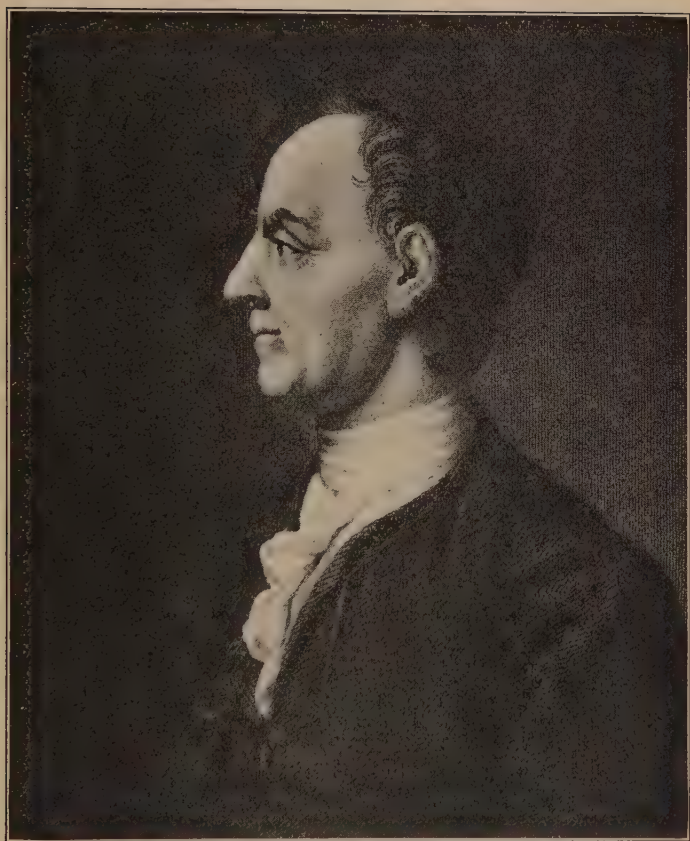
For by transposing 1 in the above equation,

$$2^n - 1 = {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots + {}_nC_n.$$

See section 70.

HISTORICAL NOTE

Sir Isaac Newton, who lived in England from 1642 to 1727, was perhaps the greatest mathematician and scientist who ever lived. His principal discoveries are contained in a monumental work



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called the "Principia," which was published by him in 1687. Among these discoveries we find the Binomial Theorem, the beginnings of the Calculus, the statement of the laws of motion, the fact of universal gravitation and the laws governing it. Newton also introduced the use of literal exponents, discovered a method of determining limits to the number of positive and negative roots in an equation, proved that imaginary roots occur in pairs in an equation with real coefficients, explained the decomposition of light and the cause of the rainbow, and invented the sextant and the reflecting telescope. He invented a theory concerning the nature of light known as the "Corpuscular Theory."

The last years of his life were marred by poor health and shattered nerves. He is buried in Westminster Abbey, where a great monument has been erected to his memory.

While Newton probably discovered that this theorem was true for positive integers as well as for negative and fractional values, Euler (1707-1783) was perhaps the first to give us a proof of the theorem, and Abel (1802-1829), a Norwegian, is probably the one to whom we are indebted for the first satisfactory proof.

Cumulative Review

1. Write the first four terms of the expansion of $(x^{-\frac{1}{2}} - 3x^2)^{13}$ and reduce the result to the simplest form.

2. Expand $(x\sqrt{y} + y\sqrt{x})^5$ and express the terms of the result in simplest form with fractional exponents.

3. Write out the first four terms of $(x - y)^8$. Find the fourth term of this expression when $x = \frac{1}{2}a^{\frac{2}{3}}b^{\frac{1}{3}}$, $y = \frac{4}{7}a^{\frac{1}{3}}b^{\frac{4}{3}}$, and reduce the result to its simplest form.

4. Without expanding, find the middle term of

$$(x + x^{-1})^{10}.$$

5. Without expanding, find the middle term of $(e^x - e^{-x})^6$.

6. If the middle term of the expansion of $\left(3x - \frac{1}{2\sqrt{x}}\right)^4$ is equal to the fourth term of the expansion of $\left(2\sqrt{x} + \frac{1}{2x}\right)^7$, find the value of x .

7. Construct a graph on the scale of 1 inch to 1000 copies and the same unit to \$100 to show the total cost of any number of copies up to 3000. Find from the figure the cost of 2500 copies and the number of copies costing \$275.

8. How many terms of the progression $\frac{5}{3}, \frac{23}{12}, \frac{13}{6}, \dots$, must be taken to make the sum $36\frac{1}{2}$?

9. The number 2 is a geometric mean between x and y , y is an arithmetic mean between $2x$ and z , and z exceeds the sum of $4x$ and y by 5. Find x , y , and z .

10. Given a square whose side is 2. The middle points of its adjacent sides are joined by straight lines forming a second square inscribed in the first. In the same manner, a third square is inscribed in the second, a fourth in the third, and so on indefinitely. Find the sum of the perimeters of all the squares.

11. Each stroke of an air pump exhausts one tenth of the air in the receiver. How much of the air originally in the receiver is left after 5 strokes?

12. The distance through which a body falls from rest varies as the square of the time. If a body falls 144 feet in the first 3 seconds, how far will it fall in the first 5 seconds?

13. What is the value of x if

$$\left(\frac{a^3 - b^3}{a - b} - ab\right) : \left(\frac{a^3 + b^3}{a + b} + ab\right) = 1 : x?$$

14. The diagonal of a rectangle is twice one side. Find the ratio of the sides.

15. Simplify $i^{20} - i^{18} + i^{16} - i^{14}$.

16. $(x - 6\sqrt{-7})^2 = ?$

17. Rationalize $\frac{2 + \sqrt{-5}}{2 - \sqrt{-5}}$.

18. Reduce to the form $a + bi$

(a) $\frac{4 + \sqrt{-3}}{2 - \sqrt{-3}}$. (b) $(2 - 3\sqrt{-4})^2$. (c) $\sqrt{5 - 12i}$.

19. How many arrangements of the first six letters of the alphabet can be made in which the two vowels shall precede the consonants?

20. Derive the formula for the number of permutations of n things taken r at a time.

21. Express the recurring decimal $.121212 \dots$ as a geometric progression and then find its value.

22. Prove that $(1 - i)^3 = \frac{-4i}{1 + i}$.

23. The area of a circle varies as the square of its radius. When the radius is 6 feet the area is 113.0976 square feet. What is the area when the radius is 8 feet?

24. A man bought two farms for \$2800 each. The larger contained 10 acres more than the smaller, but he paid \$5 more per acre for the smaller than for the larger. How many acres did each contain?

25. Solve $x + x^{-1} = 2.9$.

26. Prove that the sum of the fraction $\frac{a}{b}$ and its reciprocal is greater than 2, provided a does not equal b .

CHAPTER VIII

THEORY OF QUADRATIC EQUATIONS

THE topics of the present chapter will give you a clearer conception of the nature of a quadratic equation and of its roots. The chapter is a foundation for the following chapter on equations in general.

79. Character of the Roots. — Under the title “*character*” or “*nature*” we usually include three points with reference to the roots of a quadratic equation; namely, whether the roots are *real* or *imaginary*, and if real, whether they are *rational* or *irrational*, and whether *equal* or *unequal*.

The roots of any quadratic equation may be expressed in general form thus:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

NOTE. r_1 is read r sub one, and r_2 , r sub two.

You will see that the question whether the roots are real or imaginary depends upon whether the quantity under the radical sign is positive or negative. In fact, the whole question of the character of the roots may be determined by examination of the value of $b^2 - 4ac$ in the given equation.

The expression $b^2 - 4ac$ is called the *discriminant* of the equation.

Rule. — (1) *If the discriminant is a perfect square or 0, the roots are rational; if not, they are irrational.*

(2) *If $b^2 - 4ac$ is greater than 0, the roots are real and unequal.*

(3) *If $b^2 - 4ac$ is equal to 0, the roots are real and equal.*

(4) *If $b^2 - 4ac$ is less than 0, the roots are imaginary.*

NOTE. In applying the above rule notice that if an equation with real coefficients has imaginary roots, they *must* be unequal.

The terms *rational* and *irrational* are not usually applied to imaginary numbers.

It is understood that the equations to which these rules are applied have real and rational coefficients.

Illustrative examples :

Determine the character of the roots of each of the following equations :

- | | |
|---------------------------|------------------------------|
| 1. $2x^2 - 8x = 7.$ | 3. $3x - 2 = -\frac{1}{3x}.$ |
| 2. $3x^2 - 10x + 50 = 0.$ | 4. $7x^2 - 20x - 3 = 0.$ |

Solutions :

1. $2x^2 - 8x - 7 = 0.$ $b^2 - 4ac = 64 + 56 = 120.$

Therefore the roots are real, unequal, and irrational.

2. $3x^2 - 10x + 50 = 0.$ $b^2 - 4ac = 100 - 600 = -500.$

Therefore the roots are imaginary.

3. Clearing of fractions, we have

$$9x^2 - 6x + 1 = 0. \quad b^2 - 4ac = 36 - 36 = 0.$$

Therefore the roots are real, equal, and rational.

4. $7x^2 - 20x - 3 = 0.$ $b^2 - 4ac = 400 + 84 = 484.$

As 484 is a perfect square, the roots are real, unequal, and rational.

80. Determining Literal Coefficients. — In the equation $2x^2 - mx + 3 = 0$, you will see at once that the value of the discriminant, and hence the character of the roots, depends on the value of m .

For example, if m is 1, the roots of the above equation are imaginary, and if $m = 10$, the roots are real, unequal, and irrational. We are sometimes asked to determine the value

of such a literal coefficient so as to make the roots of the equation have any desired character.

Illustrative example I:

Determine the value of m in the equation

$$4x^2 + mx + 9 = 0,$$

so as to make the roots of the equation real, equal, and rational.

Solution: Since the roots are to be real, equal, and rational, $b^2 - 4ac$ must equal 0.

$$\begin{aligned} \text{Substituting in } b^2 - 4ac &= 0, \\ m^2 - 144 &= 0. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } m^2 &= 144, \\ \text{and } m &= \pm 12. \end{aligned}$$

Check: If $m = +12$, the equation becomes $4x^2 + 12x + 9 = 0$, and the roots are $-\frac{3}{2}$ and $-\frac{3}{2}$.

If $m = -12$, the equation becomes $4x^2 - 12x + 9 = 0$, and the roots are $\frac{3}{2}$ and $\frac{3}{2}$.

Illustrative example II:

Determine the value of m which will make the roots of the equation $(m+1)x^2 - 20x + \frac{m}{6} = 0$, real and equal.

Solution: Clearing of fractions, we obtain

$$\begin{aligned} 6(m+1)x^2 - 120x + m &= 0. \\ b^2 - 4ac &= 0. \end{aligned}$$

$$\begin{aligned} \text{Therefore, } 14,400 - 24m(m+1) &= 0. \\ m(m+1) - 600 &= 0. \\ m^2 + m - 600 &= 0. \end{aligned}$$

$$m = 24 \text{ or } -25.$$

Illustrative example III :

Determine the value of m which will make the roots of the equation $(m - 3)x^2 + 4mx + 50 = 0$ imaginary. Determine also the values of m which will make the roots real and unequal.

Solution: (a) If the roots of the above equation are to be imaginary,

$$b^2 - 4ac < 0.$$

Hence, $16m^2 - 200(m - 3) < 0.$

$$16m^2 - 200m + 600 < 0.$$

$$(m - 5)(16m - 120) < 0.$$

Since the product of these two factors < 0 , they must have opposite signs.

Hence, either $m - 5 > 0$ and $16m - 120 < 0$,
or else $m - 5 < 0$ and $16m - 120 > 0$.

In the first case $m > 5$ and $m < 7\frac{1}{2}$.

Hence, any value of m lying between 5 and $7\frac{1}{2}$ satisfies the inequality.

In the second case $m < 5$ and $> 7\frac{1}{2}$, and there are no such values of m .

Hence, the first case gives the solution.

Solution: (b) If the roots of the equation are to be real and unequal,

$$b^2 - 4ac > 0.$$

Then $(m - 5)(16m - 120) > 0.$

Since the product of these factors is positive they must have the same sign and either $m - 5 > 0$ and $16m - 120 > 0$, or else $m - 5 < 0$ and $16m - 120 < 0$. In the first case $m > 5$ and $> 7\frac{1}{2}$. Since all values of m greater than $7\frac{1}{2}$ are also greater than 5, they satisfy the inequality.

In the second case $m < 5$ and $m < 7\frac{1}{2}$. Since all values of m less than 5 are also less than $7\frac{1}{2}$, they satisfy the inequality.

We conclude that the roots are real and unequal for all values of m less than 5, and for all values greater than $7\frac{1}{2}$.

Exercises

Without solving, determine the character of the roots in each of the following equations:

1. $6x^2 + x = 1.$

10. $9x^2 - 18x + 4 = 0.$

2. $3x^2 - 5x + 1 = 0.$

11. $x^2 + x + 1 = 0.$

3. $x^2 + 5 = \frac{14x}{3}.$

12. $\frac{4x^2 + 1}{2} = -2x.$

4. $3x^2 + 7x + 5 = 0.$

13. $4 = \frac{2}{x^2} + \frac{7}{x}.$

5. $49x^2 + 1 = 14x.$

14. $3x^2 - 1 = \frac{2x - 3}{5}.$

6. $11x^2 + 3 = 12x.$

7. $9 + 16x^2 = -24x.$

15. $x^2 - 6x + 9 = 0.$

8. $3x^2 - \frac{1}{3}x = \frac{8}{3}.$

16. $x^2 - 5x + 6 = 0.$

9. $3x(3x - 2) + 5 = 0.$

17. $x^2 - 16 = 0.$

18. Check examples 15, 16, and 17 by solving by factoring.

Determine the value of m which will make the roots of each of the following equations real, equal, and rational.

19. $x^2 - 14x - m = 0.$

21. $49x^2 - 2mx + 36 = 0.$

20. $3mx^2 + 6x + 1 = 0.$

22. $(m - 1)x^2 - 6x = -1.$

23. $(m + 1)x^2 + \frac{3mx}{4} + 1 = 0.$

24. $(m + 1)x^2 + mx - 1 = 0.$

25. $2mx^2 + (m + 10)x + 9 = 0.$

26. Solve the equation $x^2 - 2x + 2 = 0$ by the formula. What do you observe about both roots? Check by the use of the discriminant.

27. What value of m in $x^2 - 4x + m$ will make the roots
(a) real and unequal?
(b) real and equal?

Determine the values of m which will make the roots real and unequal in

28. $x^2 - 2mx + 7m + 8 = 0$.
29. $x^2 - (10m + 2)x + 60m + 1 = 0$.
30. $(m + 1)x^2 + 4mx + 9 = 0$.

Determine the value of m which will make the roots imaginary in

31. $(2m + 1)^2x^2 - (m + 2)x + 1 = 0$.
32. $9x^2 - (5m - 1)x + 3m + 1 = 0$.
33. $2mx^2 + (5m + 2)x + 4m + 1 = 0$.
34. $x^2 + 4mx + 3m^2 + 4 = 0$.

For what values of m will each of the following equations have equal roots?

35. $(m + 2)x^2 - 2mx + 1 = 0$.
36. $x^2 + 2mx + 9 = 0$.
37. $x^2 - mx + (m + 3) = 0$.
38. $4x^2 + 3mx + 9 = 0$.

39. Solve $x^2 + 5x + m = 0$. What is the least integer which, when substituted for m in the equation, makes the roots of the equation imaginary?

40. Two students attempt to solve a problem that reduces to a quadratic equation. One in reducing has made a mistake only in the constant term of the equation and finds 8 and 2 for the roots. The other has made a mistake only in the coefficient of the first-degree term of the equation and finds -9 and -1 for the roots. What are the true roots and what should the equation be?

81. The Incomplete Quadratic. — When we represent any quadratic equation in the general form $ax^2 + bx + c = 0$, we understand that a , b , and c may represent any real numbers whatever. Either b or c or both may be 0, and we shall still have a quadratic equation, though consisting of one or two terms. Such quadratic equations are called *incomplete*.

If $c = 0$, the equation becomes $ax^2 + bx = 0$. x is now a factor of the equation, and hence $x = 0$ is a root.

If both b and c are 0, the equation becomes $ax^2 = 0$. x^2 is now a factor, or x is a factor twice. Hence, $x = 0$ is a root twice.

If $b = 0$, but c does not equal 0, the equation reduces to the form $ax^2 + c = 0$.

Hence, $x^2 = -\frac{c}{a}$ and $x = \pm \sqrt{-\frac{c}{a}}$. That is, the roots are

numerically equal, but opposite in sign. Does the above statement imply that these roots are necessarily imaginary? Why?

If $a = 0$, the equation ceases to be a quadratic.

We now state the following important

Principles. — (1) *One root of the general quadratic equation $ax^2 + bx + c = 0$ is 0, if, and only if, $c = 0$.*

(2) *Both roots of the general quadratic equation*

$$ax^2 + bx + c = 0$$

are 0, if, and only if, both b and c are 0.

(3) *The roots of the general quadratic equation*

$$ax^2 + bx + c = 0$$

are numerically equal but opposite in sign, if, and only if, $b = 0$.

82. Formation of Equations. — By the factor theorem, if m is a number which reduces the expression $ax^2 + bx + c$ to 0 when it is substituted for x , then $x - m$ is a factor of

$ax^2 + bx + c$. But any number which reduces $ax^2 + bx + c$ to 0 is a root of the equation $ax^2 + bx + c = 0$.

Hence, in any equation, $x - a$ root is a factor of the left member set equal to 0.

If, now, we have given two roots, say a and b , we may write the equation whose roots they are, by multiplying the factors $(x - a)$ and $(x - b)$ and setting their product equal to 0.

Rule. — *To form an equation from given roots,*

(1) *Write $x -$ each root as a factor.*

(2) *Multiply these factors together and set their product equal to 0.*

(3) *Clear of fractions if necessary.*

Illustrative example I:

Write the equation whose roots are 3 and -5 .

Solution: The corresponding factors are $x - 3$ and $x + 5$.

The equation is $(x - 3)(x + 5) = 0$,

or $x^2 + 2x - 15 = 0$.

Illustrative example II:

Write the equation whose roots are $-3 + \sqrt{-5}$, and $-3 - \sqrt{-5}$.

Solution: The corresponding factors of the equation are

$(x + 3 - \sqrt{-5})$ and $(x + 3 + \sqrt{-5})$.

The equation, therefore, is

$(x + 3 - \sqrt{-5})(x + 3 + \sqrt{-5}) = 0$,

or $x^2 + 6x + 14 = 0$.

83. Roots and Coefficients. — Since different quadratic equations have different roots, and one quadratic equation differs from another only in coefficients, you will see that the roots must be determined by the coefficients. In fact, this relation is very definite, as we shall now see.

We are accustomed to write the roots of a quadratic equation as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

This expresses both roots together. Let us separate them, denoting one root by r_1 and the other by r_2 .

$$\text{Then} \quad r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\text{Adding,} \quad r_1 + r_2 = -\frac{b}{a}$$

$$\begin{aligned} \text{Multiplying,} \quad r_1 r_2 &= \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{b^2 - b^2 + 4ac}{4a^2} \\ &= \frac{c}{a}. \end{aligned}$$

If we divide the quadratic equation $ax^2 + bx + c = 0$ by a , we obtain $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Hence the

Principle. — *After a quadratic equation in x has been divided by the coefficient of x^2 , the sum of the roots equals the coefficient of x with sign changed $\left(-\frac{b}{a}\right)$ and the product of the roots equals the constant term $\left(\frac{c}{a}\right)$.*

The above principle may also be used to write an equation from given roots.

Illustrative examples :

Write the equation whose roots are $-\frac{1}{2}$ and 3.

Solution: The equation is to have two roots, hence it will be a quadratic equation, and the first term is x^2 .

The sum of the roots is $\frac{5}{2}$. Hence the coefficient of x is $-\frac{5}{2}$.

The product of the roots is $-\frac{3}{2}$, hence the third term is $-\frac{3}{2}$.

The equation then is $x^2 - \frac{5}{2}x - \frac{3}{2} = 0$,
or $2x^2 - 5x - 3 = 0$.

This principle also enables us to solve many such examples as the following.

In the equation $x^2 + kx + 18 = 0$, one root is twice the other. Determine the value of k .

Solution: Denote the roots by r and $2r$ respectively.

Then, $-3r = k$, (1)

and $2r^2 = 18$. (2)

Hence, $r^2 = 9$,

and $r = \pm 3$.

Therefore, $k = \mp 9$. By substituting in (1).

The principle here stated is often very useful in checking the solutions of quadratic equations.

For example, determine whether $\frac{1}{2} + \frac{1}{2}\sqrt{17}$ and $\frac{1}{2} - \frac{1}{2}\sqrt{17}$ are roots of the equation $x^2 - x - 4 = 0$.

Solution: $(\frac{1}{2} + \frac{1}{2}\sqrt{17}) + (\frac{1}{2} - \frac{1}{2}\sqrt{17}) = 1$, which equals the coefficient of x with sign changed.

Also $(\frac{1}{2} + \frac{1}{2}\sqrt{17})(\frac{1}{2} - \frac{1}{2}\sqrt{17}) = \frac{1}{4} - \frac{17}{4} = -\frac{16}{4} = -4$, which is the constant term.

Hence these are the roots of the equation.

Exercises

Determine, by inspection, the sum and the product of the roots of the equations:

1. $x^2 + 4x + 4 = 0$.

Check by solving the equation, and testing the roots.

2. $x^2 - 8x + 15 = 0$. Check.

3. $x^2 + 9x + 18 = 0$. 8. $3k = 2x - x^2$.
 4. $3x^2 + 4x - 6 = 0$. 9. $x^2 + 4x + h - k = 0$.
 5. $5x^2 - 3x - 2 = 0$. 10. $x^2 - x + .5 = 0$.
 6. $2x^2 + 5ax - 6a^2 = 0$. 11. $4x + 5 = 3x^2$.
 7. $\frac{11x - 2x^2}{x + 2} = x - 2$.

Using the method of factors, write the equations whose roots are

12. 3, -1. 13. -5, -2. 14. $\frac{1}{2}, \frac{1}{3}$.
 15. $\sqrt{2}, -\sqrt{2}$. 18. $\frac{1}{2} + \frac{1}{2}\sqrt{3}, \frac{1}{2} - \frac{1}{2}\sqrt{3}$.
 16. $3 - \sqrt{2}, 3 + \sqrt{2}$. 19. $3 - \sqrt{-1}, 3 + \sqrt{-1}$.
 17. $a + b, a - b$. 20. $-1 + \sqrt{-2}, -1 - \sqrt{-2}$.

Using the relation of roots to coefficients, write the equations whose roots are

21. -3, -2. 24. $3\sqrt{2}, -3\sqrt{2}$. 27. $-\frac{1}{2}, -\frac{1}{3}$.
 22. $5, \frac{2}{3}$. 25. -4, -3. 28. $-\frac{3}{4}, -2$.
 23. -2, $-\frac{1}{2}$. 26. 1.5, 2. 29. $2a, -7a$.
 30. $-\frac{n}{m}, \frac{m}{n}$. 33. $2 + i, 2 - i$.
 31. $3 - \sqrt{3}, 3 + \sqrt{3}$. 34. $-3 + \sqrt{-1}, -3 - \sqrt{-1}$.
 32. $\frac{2}{3} + \frac{1}{3}\sqrt{2}, \frac{2}{3} - \frac{1}{3}\sqrt{2}$. 35. $-\frac{1}{2} + \sqrt{-3}, -\frac{1}{2} - \sqrt{-3}$.

Determine the value of k in each of the following:

36. $x^2 + kx - 14 = 0$ if one root is 2.
 37. $x^2 + 4x + k = 0$ if one root is three times the other.
 38. $x^2 + kx - 6 = 0$ if one root exceeds the other by 5.
 39. $x^2 + kx - 1 = 0$ if the roots differ only in sign.
 40. Are $2 + \sqrt{5}$ and $2 - \sqrt{5}$ roots of the equation

$$x^2 + 4x - 1 = 0?$$

41. Are $-3 + i$ and $-3 - i$ roots of the equation
$$x^2 + 6x - 12 = 0?$$
42. Are $2 + \frac{1}{2}\sqrt{-2}$ and $2 - \frac{1}{2}\sqrt{-2}$ roots of the equation
$$2x^2 - 8x + 9 = 0?$$
43. Determine by the principle of section 83, whether .14 and 2 are or are not roots of $x^2 - 2.14x + .28 = 0$.
44. Determine whether $-\frac{1}{4}$ and $\frac{1}{2}$ are or are not roots of the equation $x^2 - \frac{1}{8} = \frac{1}{4}x$.

Cumulative Review

1. The sum of the terms of an arithmetic progression is $28\frac{1}{2}$, the first term is -12 , the common difference is $\frac{3}{2}$. Find the number of terms in the series.

2. Find the sixth term in the expansion of

$$\left(a^2 - \frac{1}{2\sqrt{a}}\right)^{10}.$$

3. Two different prizes are to be awarded to 10 people. In how many ways may they be awarded in order that no one person shall receive both prizes?

4. Construct a price graph of gasoline when it is retailing at 25 cents a gallon. Let the x -axis denote number of gallons up to 10, and the y -axis denote 25 cents to each square or unit. From your chart find the cost of 7 gallons of gasoline.

5. Prove, without substituting numbers for the letters, which is the greater $m^2 + m$ or $m^3 + 1$.

6. Reduce $\frac{11 - 3i}{1 - 3i}$ and $\frac{2i - 9}{4 + i}$ to simplest form and construct their difference graphically, if $i = \sqrt{-1}$.

7. A line a feet long is divided into two segments so that the longer segment is a mean proportional between the whole

line and the shorter segment. Find the length of the two segments and interpret the two results.

8. Solve
$$\begin{aligned}x^3 - y^3 &= 1304. \\x^2 + xy + y^2 &= 163.\end{aligned}$$

9. Solve the equation

$$(m - 4)x^2 + 4(m + 1)x - 4(m + 1) = 0,$$

for x , and find for what values of m the roots are equal. For what values of m are the roots real? For what imaginary?

10. What are the values of b and c in the quadratic equation $x^2 + bx + c = 0$

(a) If the roots are 6 and -10 ?

(b) If one of the roots is $3 - \sqrt{5}$?

11. Show that, for real values of x , $9x^2 + 25$ cannot be less than $30x$.

12. How many games must be played in a league of ten baseball clubs if each club plays 10 games with every other club?

13. Simplify

(a) i^7 .	(c) i^{41} .	(e) i^{90} .
(b) i^{14} .	(d) i^{57} .	(f) i^{125} .

14. Compute the smaller of the two roots of the equation

$$x^2 - 0.802x + 0.104 = 0$$

correct to three significant figures.

15. If one of the roots of the equation

$$ax^2 + bx + c = 0$$

is double the other, show that $9ac = 2b^2$.

16. A train A starts to go from P to Q , two stations 240 miles apart, and travels uniformly. One hour later another train B starts from P , and after traveling for 2 hours, comes to a point that A had passed 45 minutes previously. The

speed of B is now increased by 5 miles an hour and it overtakes A just on entering Q . Find the rates at which they started.

17. Solve $mx - y = 1$, $3(x - 1)^2 + y^2 = 1$ simultaneously. For what values of m are the two roots equal?

18. Prove

$$(a) \ a^0 = 1. \qquad (b) \ x = a^{\frac{1}{p}} \text{ if } x^p = a.$$

19. Find the square root of $44 - 7\sqrt{-20}$.

20. Given $a = 1$, $n = 6$, $S = 27$, find d and the first six terms of the arithmetic progression.

21. The length of each swing of a certain pendulum is $\frac{9}{10}$ of the length of the preceding swing. If the first swing is 6 inches long, show that the total distance through which the pendulum moves cannot be greater than 5 feet.

22. A goes from P to Q in 14 hours, B starts at the same time from a point 10 miles behind P and arrives at Q at the same time as A. B finds that he takes a half hour less to go 20 miles than A does. Find the distance between P and Q .

23. It is required that in the equation $(x + 1)^2 = 6x - k$ the values of x shall be real. How large may k be?

24. A regular octagon is formed by cutting off the corners of a square whose edge is 1 foot. Find the side of the octagon.

25. The illumination from a source of light is inversely proportional to the square of the distance from the source. If my book is 8 feet from the lamp, how near must I move it in order that the light on it shall be doubled?

CHAPTER IX

GENERAL THEORY OF EQUATIONS

SINCE applications of algebra to higher mathematics and to applied problems are usually in the form of equations, the ability to solve equations of all types becomes necessary. The present chapter considers equations in general with special reference to those of higher degree than the second. The chapter also contains some discussions which are intended to make clearer the real nature and meaning of algebraic equations.

84. Functions of x . — Any expression whose value depends solely upon the value of the variable x is a function of x . The symbol used to represent any function of x is $f(x)$, which is read “function of x .” Any function of x in which the variable x is operated upon only by the fundamental processes of addition, subtraction, multiplication, division, involution, and evolution is an *algebraic function*, otherwise it is a *transcendental function*.

Examples of transcendental functions are $\log x$, $\sin x$, e^x , etc.

An *integral* function of x does not contain x either in the denominator of a fraction or with a negative exponent.

A *rational* function of x does not contain x either under a radical sign or with a fractional exponent.

85. Algebraic Equations. — An *algebraic equation* is formed when an algebraic function is set equal to zero or other constant, or another algebraic function.

The equations considered in this chapter are integral and rational, and except as otherwise stated have real and rational coefficients.

86. General Forms of the Algebraic Equations. — Every algebraic equation can be reduced by the use of axioms to the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0, \quad (1)$$

which may therefore be regarded as the type of any algebraic equation. Dividing every term of this equation by a_0 and substituting p_1 for $\frac{a_1}{a_0}$, etc., we obtain the equation of the n th

degree in its simplest general form :

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0. \quad (2)$$

When we are considering a given function of x , the result obtained by substituting a for x is called $f(a)$.

Illustrative example I :

If $f(x) = x^3 - 5x^2 - 2x + 24$, find the value of $f(2)$. Of $f(-1)$. Of $f(4)$.

$$\text{Solutions:} \quad f(2) = 2^3 - 5(2)^2 - 2 \cdot 2 + 24 = +8.$$

$$f(-1) = (-1)^3 - 5(-1)^2 - 2(-1) + 24 = 20.$$

$$f(4) = 4^3 - 5(4)^2 - 2(4) + 24 = 0.$$

Illustrative example II :

Change the equation $\frac{1}{2}x^4 - \frac{3}{4}x^3 - \frac{2}{3}x^2 = \frac{1}{8}$ to both forms (1) and (2).

Solution: Multiplying every term by 24 and then transposing, we have

$$12x^4 - 18x^3 - 16x^2 - 3 = 0, \text{ which is form (1).}$$

Now dividing every term by 12 we obtain

$$x^4 - \frac{18}{12}x^3 - \frac{16}{12}x^2 - \frac{3}{12} = 0,$$

or
$$x^4 - \frac{3}{2}x^3 - \frac{4}{3}x^2 - \frac{1}{4} = 0, \text{ which is form (2).}$$

Exercises

1. Given $f(x) = 2x^3 - 3x^2 + x + 2$:
Find $f(1)$, $f(2)$, $f(\frac{1}{2})$, $f(-1)$, $f(0)$.
2. If $f(x) = x^3 - 3x + 2$:
Find $f(0)$, $f(1)$, $f(-1)$.
3. If $f(x) = x^2 + x$:
Find $f(1)$, $f(0)$, $f(2x)$.
4. If $f(x) = 3x^3 - 2x^2 - 5x + 4$:
Find $f(0)$, $f(1)$, $f(2)$, $f(-1)$, $f(a)$, $f(-x)$.

Reduce the following equations to forms (1) and (2) above:

5. $x - \frac{x^2}{2} + \frac{x^3}{4} = 1 + \frac{3x}{8}$.

6. $\frac{4x^5}{7} + 2 = \frac{3x^3}{14} + 3x$.

7. $.4x^3 = .8x^2 - .16x + .24$.

8. $3.6x^5 - 2.7x^2 = 1.8x^4 + .9x$.

9. Change the equation $.5x^2 + .25x = 9$ to form (1) and then solve it.

10. What is the trinomial in x and of the second degree, whose graph passes through the point $(1, -2)$, and also such that the equation formed when setting it equal to zero gives roots of 3 and 2?

You are to understand that the coefficients $p_1, p_2, \dots, p_n$ may be positive or negative, integral or fractional, rational or irrational, real or imaginary, or zero. If any one of these coefficients is zero, the equation is said to be *incomplete*.

Throughout this chapter we shall use either of the above forms to represent the general equation, as seems most convenient.

87. Distinction between a Function and an Equation. —

You should clearly recognize the fact that a function is not an equation, but that an equation is formed when the function is set equal to a constant. Clearly, no axiom can be applied to a function, since it has only one member. Hence, any such operation as changing the signs of the terms of the function, multiplying it by a constant to remove fractional coefficients, etc., cannot legitimately be performed, since the value of the function is thereby changed. We may, however, factor it, reduce fractions to a common denominator, or make such other changes in form as will not change the value of the expression.

$$\begin{array}{ll} \text{For example} & 2x^2 - 2x - 12 = 2(x-3)(x+2) \\ \text{but} & 2x^2 - 2x - 12 \neq x^2 - x + 6. \end{array}$$

$$\text{Also} \quad \frac{x^2}{3} + \frac{x}{5} = \frac{5x^2 + 3x}{15}$$

$$\text{but} \quad \frac{x^2}{3} + \frac{x}{5} \neq 5x^2 + 3x.$$

$$\text{And} \quad -x^2 - 10 = -(x^2 + 10) \text{ but } -x^2 - 10 \neq x^2 + 10.$$

88. Function Graphs.—The graph shows clearly the effects on a function of the operations mentioned above.

Let us consider the graph of $f(x) = x^2 - 2x - 8$.

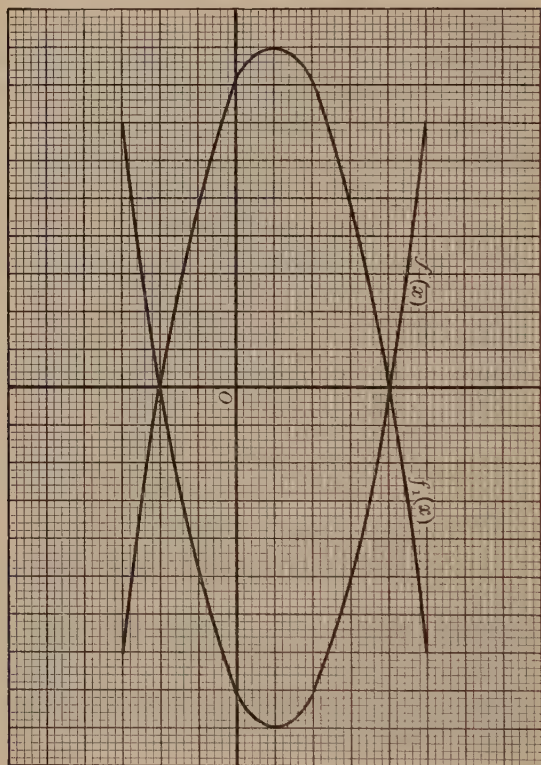
We have the following table of values :

x	1	2	3	4	5	0	-1	-2	-3
$f(x)$	-9	-8	-5	0	7	-8	-5	0	7

For $f_1(x) = -x^2 + 2x + 8$ we have the following values :

x	1	2	3	4	5	0	-1	-2	-3
$f_1(x)$	9	8	5	0	-7	8	5	0	-7

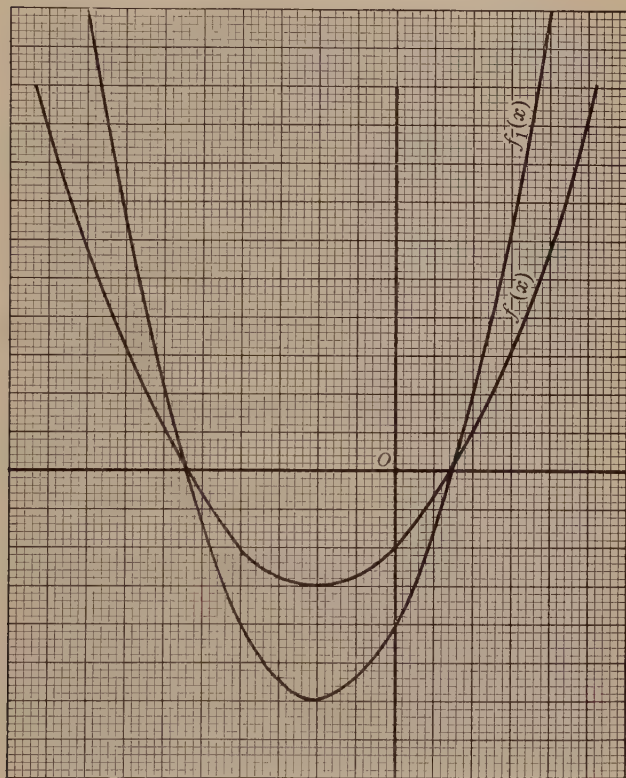
The graphs follow :



You should not fail to notice that corresponding to any given value of x , the ordinates, which represent the corresponding values of $f(x)$, are numerically equal but opposite in sign. The only points which the graphs have in common are those in which the function equals 0, for $-0 = +0$, and it is therefore only at these particular points that the sign change does not affect the value of the function.

Similarly, we shall compare the graphs of $f(x) = \frac{1}{2}x^2 + x - 1$, and $f_1(x) = x^2 + 2x - 2$.

The graphs follow :



In this case $f_1(x)$ is obtained by multiplying $f(x)$ by 2.

Hence, every ordinate on the graph of $f_1(x)$ is twice the corresponding ordinate on the graph of $f(x)$. The only points which the graphs have in common are, as before, the points where the function is 0, for $2 \cdot 0 = 0$, the value of $f_1(x)$ being everywhere else different from that of $f(x)$.

Exercises

1. Plot the graph of the function $x^2 + 2x - 5$. Also plot the graph of the function $-x^2 - 2x + 5$, and compare your results.

2. Draw the graphs of the function $x^2 - 4x + 3$, and also of $-x^2 + 4x - 3$ and compare results.

3. Construct the graphs of the function $-2x^2 + 3x + 9$ and also of the function $2x^2 - 3x - 9$, and compare your results.

4. Represent graphically $f(x) = 2x^2 - x - 6$ and also $f_1(x) = -2x^2 + x + 6$.

5. Plot $f(x) = x^2 - 7x + 10$ and $f_1(x) = -x^2 + 7x - 10$.

6. Graph the function $x^2 + x - 6$. Also graph $f_1(x) = 2x^2 + 2x - 12$. Compare your results.

7. Represent graphically $f(x) = x^2 - 4x + 3$ and also $f_1(x) = 2x^2 - 8x + 6$.

8. Plot the graph of $f(x) = \frac{1}{2}x^2 - 4x + 4$ and also $f_1(x) = x^2 - 8x + 8$ and compare results.

89. The Remainder Theorem. — At this time we need to prove the following

Theorem. — *If a function of x is divided by a divisor of the form of $x - b$, the remainder is the same function of b .*

If $x^2 - x + 5$ is divided by $x - 2$, $x + 1$ is the quotient and 7 is the remainder. This may be written as follows:

$$\frac{x^2 - x + 5}{x - 2} = (x + 1) + \frac{7}{x - 2}.$$

Proof:

In general, let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$. Divide $f(x)$ by $x - b$. Let Q = the quotient, and R = the remainder.

Then
$$\frac{a_0x^n + a_1x^{n-1} + \cdots + a_n}{x - b} = Q + \frac{R}{x - b}.$$

Hence, $a_0x^n + a_1x^{n-1} + \cdots + a_n \equiv Q(x-b) + R.$

In this identity, let $x = b.$

Then $a_0b^n + a_1b^{n-1} + a_2b^{n-2} + \cdots + a_n = 0 \cdot Q + R.$

That is, $R = f(b)$, where $f(b)$ represents $f(x)$ with x replaced by $b.$

Illustrative example :

If $f(x) = 2x^3 - 6x^2 + 5x - 2$, and this is divided by $x - 3$, we shall find $f(3) = 2 \cdot 27 - 6 \cdot 9 + 5 \cdot 3 - 2 = 13.$

Now if $2x^3 - 6x^2 + 5x - 2$ is actually divided by $x - 3$, we shall obtain 13 as our remainder, and thus we find that the remainder is equal to the original function with 3 substituted for $x.$

How would you find the remainder if $f(x)$ is divided by $x + 3$?

Hint: $x + 3 = x - (-3).$

Corollary. — *If in the application of the above theorem $f(b)$ is found to be 0, then there is no remainder, and $x - b$ is an exact divisor or factor of $f(x).$*

This corollary is called the *Factor Theorem* and is familiar to you from your work in factoring.

Exercises

1. Find the remainder when $x^3 + 4x^2 - 4x + 4$ is divided by $x - 4.$

2. What is the remainder when $x^6 + 2x^2 - 4x + 1$ is divided by $x - 1$?

3. If $x^3 - 3x^2 + 15x - 20$ is divided by $x - 3$, what is the remainder?

4. Find the remainder if $x^4 + 4x^3 - 12x^2 + 1$ is divided by $x + 2$.

5. If $x^3 - 5x^2 + 15x - 75$ is divided by $x - 5$, what is the remainder? What does this prove?

6. If $(x + 4)^2 + (x + 3)^3 + (x + 2)^4$ is divided by $x + 1$, what is the remainder?

90. Number of Roots of an Equation. — In discussing this topic it will be necessary for us to assume a fundamental theorem, that every rational, integral equation has at least one root. The proof of this theorem is found in higher mathematics. Assuming this proposition, we now state the

Theorem. — *Every rational, integral, algebraic equation of the n th degree has n roots.*

Proof: Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0.$$

By the fundamental theorem above stated this equation must have one root.

Let us denote this root by r_1 .

Then by the factor theorem, $x - r_1$ is a factor of the left member of the equation.

Hence the equation reduces to $(x - r_1)f_1(x) = 0$, where $f_1(x)$ is a factor of the $(n - 1)$ degree.

But, $f_1(x)$ is a rational, integral function of the $(n - 1)$ degree.

Hence the equation $f_1(x) = 0$ has a root. Let us denote it by r_2 , then in a similar manner as before we obtain

$$f_1(x) = (x - r_2)f_2(x),$$

where $f_2(x)$ is of the $(n - 2)$ degree.

Therefore, $f(x) = (x - r_1)(x - r_2)f_2(x) = 0$.

We note that each time a factor of the form $x - r$ is removed, the degree of the resulting equation is reduced by one. Hence, by a continuation of the above process the equation may be reduced to the form

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0.$$

Hence, the equation $f(x)$ has n roots, for $f(x)$ vanishes when x is equal to any of the values $r_1, r_2, r_3, \cdots r_n$.

You will note that in the above proof nothing is said about the character of the roots. They may be rational or irrational, real or imaginary, equal or unequal.

You should also understand that if $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, it can be shown that $f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n)$.

Corollary I. — *An equation of the n th degree can have no more than n roots, for if it did it would have more than n factors, each of the first degree, since there is a factor corresponding to every root. Their product would then be of more than the n th degree, which contradicts the hypothesis.*

Corollary II. — *If an equation of the n th degree has more than n roots, it is an identity and satisfied by every value of x , in which case every coefficient is 0, for otherwise it could have only n roots by corollary I.*

Corollary III. — *Every function of the n th degree may be resolved into n linear factors of some kind.*

91. Synthetic Division. — Since it becomes necessary very frequently to divide a polynomial by a divisor of the form of $x - a$, we shall here present a short method of performing this operation known as *Horner's Method of Synthetic Division*. This is a contraction of the method of division by *detached coefficients*. (See authors' Second Course in Algebra,

pp. 392-393.) The operation may be performed as in the following

Illustrative example I:

Divide $x^3 + 4x^2 + 3x - 1$ by $x - 3$.

$$\begin{array}{r} \text{Solution:} \quad 1 \quad 4 \quad 3 \quad -1 \quad | \quad 3 \\ \quad \quad \quad 3 \quad 21 \quad 72 \\ \hline \quad \quad 1 \quad 7 \quad 24 \quad 71 \end{array}$$

The quotient is $x^2 + 7x + 24$ with a remainder of 71.

Explanation: The terms of the dividend are written down in order, using only detached coefficients. In the divisor the x term is dropped and the second term is written with opposite sign. We then draw a line under the dividend, leaving room for a row of figures above it. The first term, 1, is next brought down. Then this 1 is multiplied by the 3 in the divisor, and the resulting 3 is written under the 4. Add. The resulting 7 is multiplied by the 3 and the product written under the 3 of the dividend. Again add. We continue by the same process to the end of the line. The last sum (71) is the remainder and the other numbers are the coefficients of the quotient in order. The quotient must be of degree one lower than the dividend.

Illustrative example II:

Divide $x^4 + 6x^2 + 2x - 3$ by $x + 1$.

$$\begin{array}{r} \text{Solution:} \quad 1 \quad 0 \quad 6 \quad 2 \quad -3 \quad | \quad -1 \\ \quad \quad -1 \quad 1 \quad -7 \quad 5 \\ \hline \quad 1 \quad -1 \quad 7 \quad -5 \quad 2 \end{array}$$

The quotient is $x^3 - x^2 + 7x - 5$ and the remainder is 2.

NOTE. If any power of x is lacking in the equation, it must be replaced by the coefficient 0.

From the above work we obtain the following

Rule. — (1) *Write the coefficients of x on a line, making sure to represent any missing powers of x by zero coefficients; and bring down the first coefficient.*

(2) *Multiply the first coefficient by a , and add the product to the second coefficient.*

(3) *Multiply this resulting sum by a , and add the product to the next coefficient, and so on.*

(4) *The last sum is the remainder, and the preceding numbers in order represent the coefficients of the quotient.*

Exercises

By synthetic division, find the quotient and the remainder in each of the following :

1. $(x^3 - 2x^2 - 19x + 20) \div (x - 5)$.
2. $(x^3 + 6x^2 - 12x + 8) \div (x + 2)$.
3. $(x^3 + x^2 - 4x + 6) \div (x + 3)$.
4. $(x^4 - 4x^2 - 5x + 10) \div (x - 2)$.
5. $(x^3 + 7x^2 + 15x + 25) \div (x + 5)$.
6. $(x^4 - x^3 - 6x^2 + x - 3) \div (x - 3)$.
7. $(125x^3 - 8) \div (x - \frac{2}{5})$.
8. $(3x^4 - 2x^2 + 5x - 7) \div (x - 4)$.
9. $(x^6 - 12) \div (x - 1)$.
10. $(x^6 + 3x^2 - 9) \div (x + 2)$.

92. Depressed Equations. — If a root of an equation is known, it may be removed from the equation by dividing the equation by the corresponding factor. The quotient set equal to 0 is called a *depressed equation*. The roots of the depressed equation are the remaining roots of the given equation unchanged in value. If two or more roots are

known, the equation may be depressed by dividing by the corresponding factors separately, or by their product.

Illustrative example :

One root of the equation $x^3 + x^2 - 11x + 10 = 0$ is 2. Find the other roots.

Solution: Divide $x^3 + x^2 - 11x + 10$ by $x - 2$, using Horner's method of synthetic division.

$$\begin{array}{r} \text{Thus} \qquad \qquad \qquad 1 \quad 1 \quad -11 \quad 10 \big| 2 \\ \qquad \qquad \qquad \qquad \quad 2 \qquad \quad 6 \quad -10 \\ \hline \qquad \qquad \qquad 1 \quad 3 \quad -5 \quad 0 \end{array}$$

The quotient is $x^2 + 3x - 5$, and the depressed equation is

$$x^2 + 3x - 5 = 0.$$

Solution of this equation gives

$$x = \frac{-3 \pm \sqrt{9 + 20}}{2}.$$

Hence the roots of the original equation are 2 and $\frac{-3 \pm \sqrt{29}}{2}$.

Exercises

By the use of the depressed-equation method, find the roots of the following equations :

1. $x^3 - 37x + 84 = 0$, when one root is 3.
2. $x^3 - 7x^2 + 7x + 15 = 0$, one root being + 3.
3. $x^3 - 3x^2 - 18x + 40 = 0$, when one root is - 4.
4. $x^3 + 4x^2 - 11x - 30 = 0$, one root being - 2.
5. One root of $x^3 + 6x^2 - 6x - 63 = 0$ is 3. Find the others.
6. Two of the roots of $x^4 - 15x^2 + 10x + 24 = 0$ are 2 and 3. Find the other roots.

7. $2x^4 - x^3 - 9x^2 + 13x - 5 = 0$ has two roots, $+1$ and $+1$. Find the others.

8. $32x^3 - 32x^2 - 94x + 39 = 0$ has one root, $-\frac{3}{2}$. Find the others.

9. Find whether -7 is a root of

$$x^3 + 14x^2 + 65x + 112 = 0.$$

10. If $\frac{2}{3}$ is a root of $x^3 - \frac{1}{6}x^2 - \frac{1}{12}x - \frac{1}{6} = 0$, find the other roots.

93. Formation of Equations. — Since the left member of an equation is a product of the form $(x - r_1)(x - r_2) \cdots (x - r_n) = 0$, where $r_1, r_2, \cdots r_n$ are the roots, it follows that we may at once write the equation if the roots are known.

Illustrative example :

Write the equation whose roots are $1, -1, 3 + \sqrt{2}, 3 - \sqrt{2}$.

Solution :

$$(x - 1)(x + 1)(x - 3 - \sqrt{2})(x - 3 + \sqrt{2}) = 0,$$

or

$$x^4 - 6x^3 + 6x^2 + 6x - 7 = 0.$$

Exercises

Write the equations whose roots are

1. $3, -2, 4.$

2. $-1, -1, 3, 5.$

3. $\sqrt{2}, -\sqrt{2}, 3, -4.$

4. $\sqrt{-1}, -\sqrt{-1}, 2 + \sqrt{3}, 2 - \sqrt{3}.$

5. $(3 + 2i), (3 - 2i), 2, -1.$

6. $a, b, c.$

7. $\pm 3, \pm \sqrt{3}.$

8. How many roots has the equation $x = \sqrt[5]{1}$?

By use of the remainder theorem, find the remainder when

9. $x^3 - 6x^2 + 8x - 3$ is divided by $(x - 1)$. By $(x + 1)$. By $(x + 2)$. Which of these binomials is a factor of the expression?

10. $3x^3 + 6x^2 - 8x + 1$ is divided by $(x - 3)$. By $(x + \frac{1}{2})$.

11. $x^4 + 6x^2 - 10$ is divided by $(x - 2)$. By $(x + .3)$. By $(x - \frac{1}{3})$.

By Horner's method, find the quotient and remainder when

12. $3x^3 - 2x^2 - 4x + 1$ is divided by $(x - 2)$. By $(x + 1)$.

13. $2x^4 - 6x^3 + 5$ is divided by $(x - 1)$. By $(x - 5)$.

14. When $x^6 - 1$ is divided by $(x - 1)$. By $(x + 1)$.

15. Find the remainder when $x^7 - 1$ is divided by $(x + 1)$, using the remainder theorem. Verify by Horner's method. Which method is easier in this case?

16. Is $x^5 + 3x^3 + x^2 - 5$ exactly divisible by $(x + 1)$? By $(x - 1)$? By $(x + 5)$? By $(x - 5)$?

17. By means of the factor theorem, find the factors of

$$x^4 - x^3 + 5x^2 + x - 6.$$

18. Factor into 5 factors (not necessarily real or rational)

$$x^5 - x^4 + 2x^3 - 2x^2 - 63x + 63.$$

Solve each of the following equations :

19. $x^3 + 6x^2 - 5x - 2 = 0$, knowing that one root is 1.

20. $x^3 + 4x^2 - 11x - 30 = 0$, knowing that one root is -5 .

21. $x^3 - 4x^2 + 2x + 3 = 0$, knowing that one root is 3.

22. $x^4 + x^3 - 15x^2 - 3x + 36 = 0$, knowing that 3 and -4 are roots.

23. $2x^4 + x^3 - 12x^2 - 3x + 18 = 0$, if $\sqrt{3}$ and $-\sqrt{3}$ are roots.

24. Graph the functions

$$2x^2 - 6x + 4, x^2 - 3x + 2, \frac{1}{2}x^2 - \frac{3}{2}x + 1,$$

and $-2x^2 + 6x - 4$ and then discuss the relation of the graphs.

94. Complex Roots. —

Theorem. — *In an equation with real coefficients complex roots occur in conjugate pairs; that is, if $a + bi$ is a root of the equation, then $a - bi$ is also a root.*

Proof: Let $a + bi$ be a root of the general equation

$$f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0.$$

Then it must satisfy the equation and therefore $f(a + bi) = 0$. But $f(a + bi)$ involves various powers of i from the zero power to the n th power. However, all powers of i when simplified reduce to ± 1 or $\pm i$; hence after having been simplified, only the first power of i remains.

$f(a + bi)$, therefore, takes the form $A + Bi$, where A represents the polynomial formed from the terms not containing i , and Bi represents the polynomial containing the first power of i .

$$\therefore A + Bi = 0.$$

Then $A = 0$ and $B = 0$. (Section 47.)

Now $f(a - bi)$ differs from $f(a + bi)$ only in the sign of the odd powers of i . As these are the terms which constitute the polynomial Bi ,

$$f(a - bi) = A - Bi,$$

where A and B have the same meaning as before.

But since we have already proved that $A = 0$, and $B = 0$, $A - Bi = 0$; hence, since $f(a - bi) = 0$, $a - bi$ is a root of the equation.

Corollary. — *Every equation of odd degree must have at least one real root.*

95. Quadratic Surd Roots. —

Theorem. — *In an equation with rational coefficients quadratic surd roots occur in conjugate pairs ; that is, if $a + \sqrt{b}$ is a root of the equation, then $a - \sqrt{b}$ is also a root.*

The proof of this theorem is similar to that of the preceding section and is left as an exercise for the student.

Exercises

1. Solve the equation $x^4 + x^3 - x^2 - 2x - 2 = 0$ if one root is $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$.

2. Solve the equation $x^4 - 6x^3 + 14x^2 - 14x + 5 = 0$ if one root is $2 - i$.

3. Solve the equation $x^4 - 6x^3 - 5x^2 + 54x - 36 = 0$ if one root is $3 - \sqrt{5}$.

4. Solve the equation $x^6 - 2x^5 + 5x^4 - 2x^3 - 8x^2 + 24x - 48 = 0$ if $\sqrt{3}$ and $-2i$ are known to be roots.

5. Solve $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, given that one root is $2 - \sqrt{3}$.

6. Solve $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + i$.

7. Solve $x^3 - 9x + 28 = 0$, one root being $2 + \sqrt{-3}$.

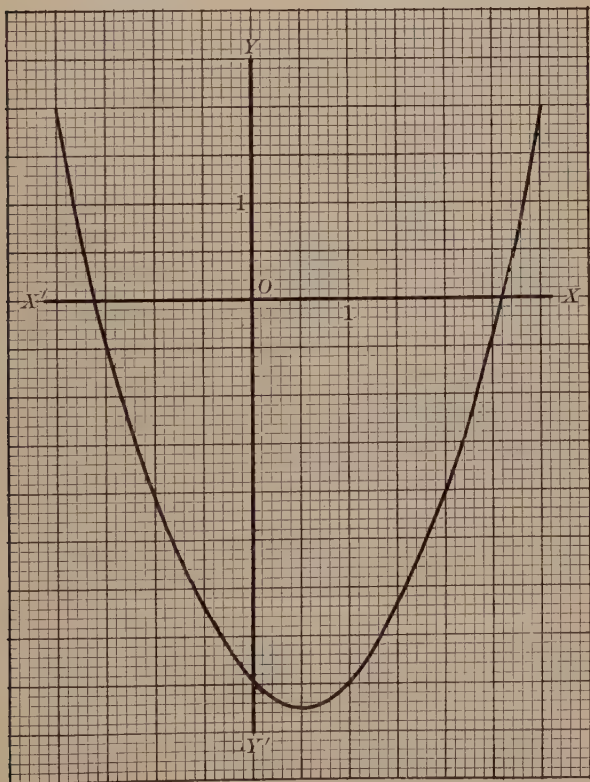
8. Solve $x^4 - 6x^3 + 24x - 16 = 0$, one root being $3 - \sqrt{5}$.

9. It is given that $3 - i$ is a root of the equation $x^3 - 8x^2 + 22x - 20 = 0$. Find the factors of the left member of this equation.

10. Write the equation of lowest possible degree with real and rational coefficients, two of whose roots are $3 - \sqrt{-1}$ and $2 + \sqrt{5}$. Write the equation if the coefficients must be real but are not required to be rational. Write it if the coefficients are not required to be either real or rational.

96. **Graphic Solution of Equations.** — Let it be required to solve by a graph the equation

$$x^2 - x - 4 = 0.$$



Graph of $x^2 - x - 4$

We first graph the function $x^2 - x - 4$, which will be a parabola as every quadratic function is. For this purpose it is best to use graph paper having the large squares subdivided into tenths or fifths. The large squares should be used as the units of the graph so that it is possible to read the result accurately to tenths.

We understand that the values of the function for successive values of x are represented by the ordinates corresponding to these values, that these are negative where the graph is below the x -axis, positive where it is above the x -axis, and 0 when it is on the axis.

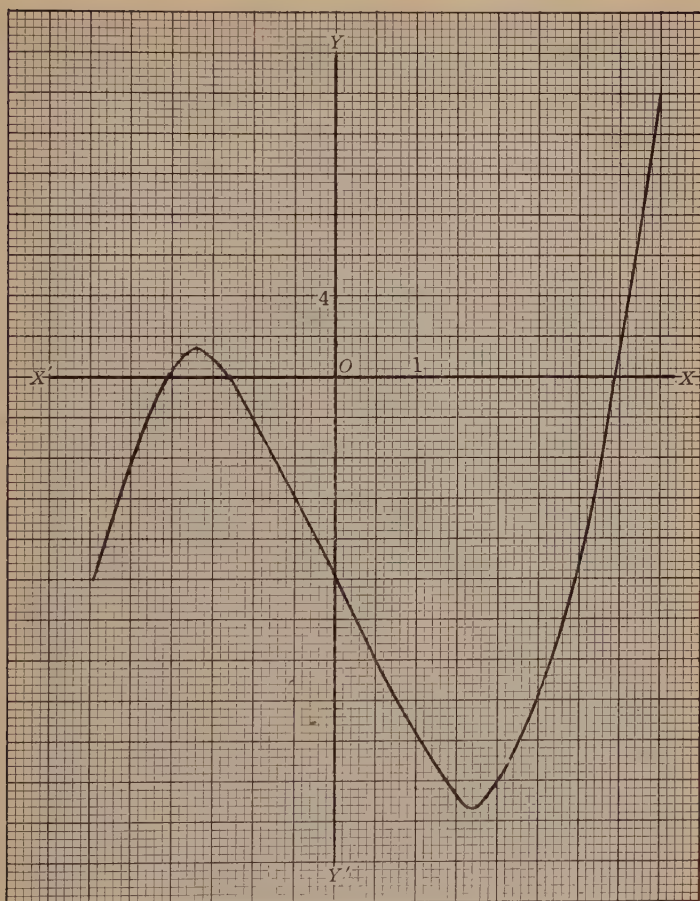
Since we are asked to find the values of x which make the function $x^2 - x - 4 = 0$, we have only to read the values of x at the point where the graph crosses the x -axis. The graph indicates roots which are approximately 2.6 and -1.6 .

Note that a graphic solution of an equation gives only approximate values for the roots. For most purposes in your algebraic work, exact answers are required, and hence you should never give a graphic solution unless the problem specifically so directs. In many practical applications, however, approximate solutions, or at least solutions in decimal form, are required, and it is here that the use of graphs proves extremely advantageous. With care, it is possible to read a well-constructed graph to hundredths with fair accuracy. For this purpose the scale must be larger than in these illustrations, and the curve made thinner.

Similarly we may solve the cubic equation

$$x^3 - 9x - 10 = 0.$$

The graph shows the roots to be -2 and approximately 3.4 and -1.3 .

Graph of $x^3 - 9x - 10$

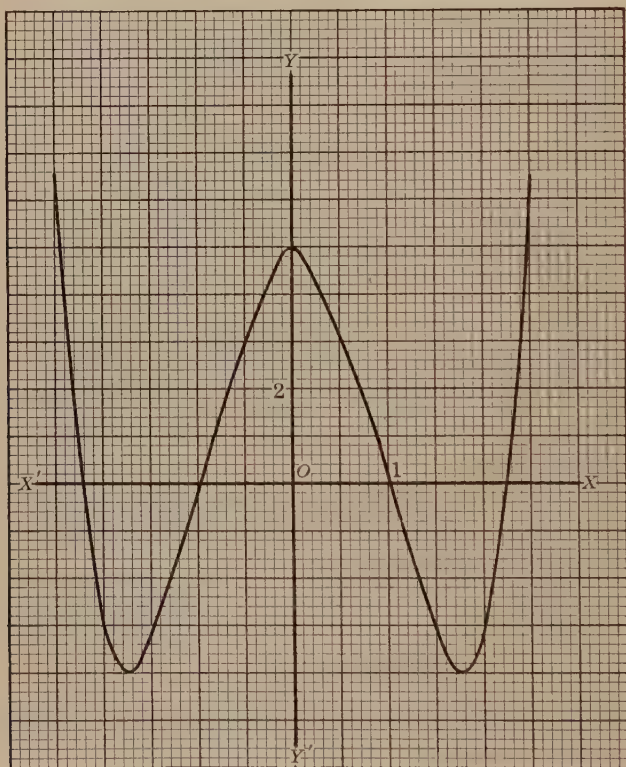
The graph shown on this page has the characteristic shape of the graph of a cubic function. The number of turning points or elbows of a graph equals, in general, one less than the degree of the function represented. The maximum number of times the graph can cross the x -axis is

equal to the degree of the function. Why? In drawing the graph we should be careful to extend the table of values far enough to include all possible intersections of the graph with the x -axis.

The graphic solution of the equation

$$x^4 - 6x^2 + 5 = 0$$

follows :



Graph of $x^4 - 6x^2 + 5$

The shape is characteristic of the graph of a quartic (or bi-quadratic) function. The roots are ± 1 and ± 2.2 .

97. Table of Values Obtained by Synthetic Division. —

In forming your table of values for the graph, it is sometimes easier to obtain the values of the function corresponding to values 1, 2, 3, etc., of x , by dividing by $(x - 1)$, $(x - 2)$, $(x - 3)$, etc., using Horner's method, and taking the remainders, rather than by substitution. Let the pupil explain why the result is the same by either process.

Exercises

Graph the following functions :

1. $x^2 + x - 6.$

4. $3x^2 - 2x - 2.$

2. $x^2 + 3x - 10.$

5. $x^3 - 5x + 3.$

3. $4x^2 + 4x + 1.$

6. $x^3 - 2x - 4.$

7. By setting each of the above functions equal to zero, find the roots of each of these equations graphically.

8. If $x = 1$ and -2 , construct a graph of the depressed equation and determine from your graph the other roots of

$$x^4 + 2x^3 - 13x^2 - 14x + 24 = 0.$$

9. If two of the roots of $x^4 + 2x^3 - 13x^2 - 38x - 24 = 0$ are 4 and -1 , find the other roots by the aid of a graph of the depressed equation.

10. Construct a graph of each of the following equations :

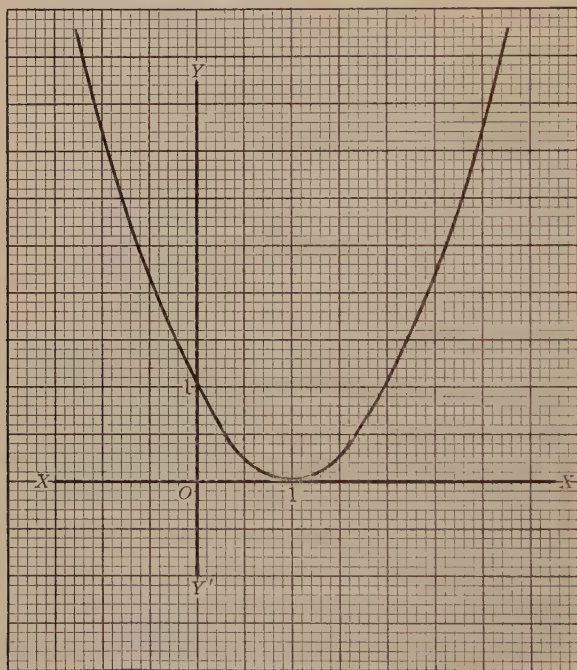
$$x^3 - 3x^2 + 2 = 0,$$

$$x^3 - 3x^2 + 4 = 0,$$

$$x^3 - 3x^2 + 6 = 0,$$

and from your graphs, obtain to the nearest tenth the roots of each of these equations. How do your curves compare in shape? How many elbows, which do not cross the x -axis, do you find in the last graph? Is there one or more pairs of imaginary roots in the last equation? What conclusion can you form from this chart?

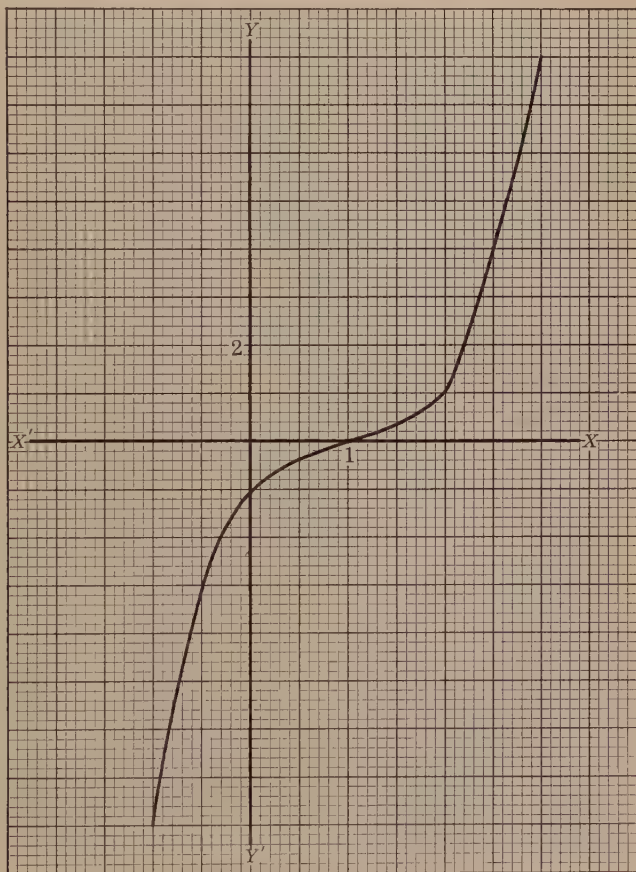
98. Multiple Roots Shown by Graph.—Two or more equal roots of an equation are called *multiple roots*. If the roots of a quadratic equation are .5 and 1.5, the graph will



Graph of $x^2 - 2x + 1$

plainly cut the axis at two points which are 1 unit apart. If, however, the roots were .9 and 1.1, the points of intersection would be only .2 of a unit apart. As the roots approach each other, the points of intersection approach each other, so that if the roots are 1 and 1, the graph is tangent to the x -axis at the point where $x = 1$. Hence a point of tangency represents two equal roots at that point.

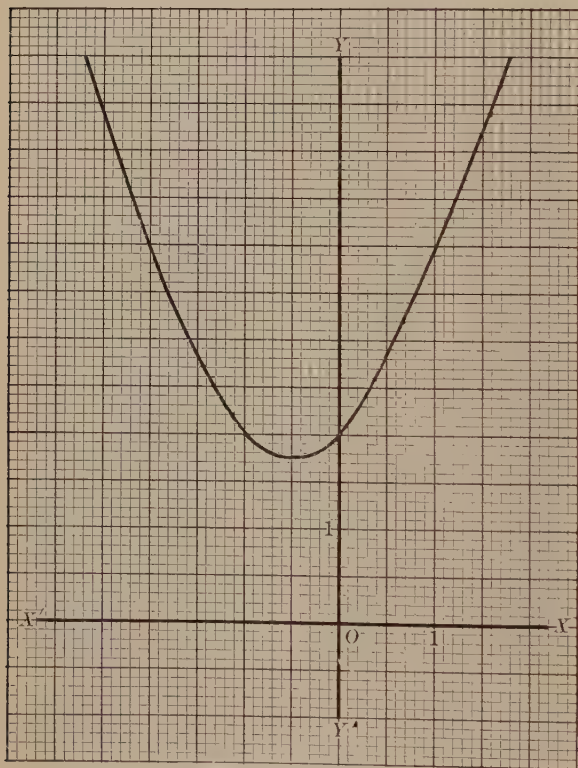
Similarly graphing $x^3 - 3x^2 + 2x = 0$, we obtain three points of intersection at 0, 1, and 2. If these roots are $\frac{1}{2}$, 1, $\frac{3}{2}$, the points of intersection will be nearer together. If the



Graph of $x^3 - 3x^2 + 3x - 1$

roots are 1, 1, 1, the three points of intersection coincide in a *point of inflection*. Hence a point of inflection occurring on the x -axis represents 3 equal roots at that point.

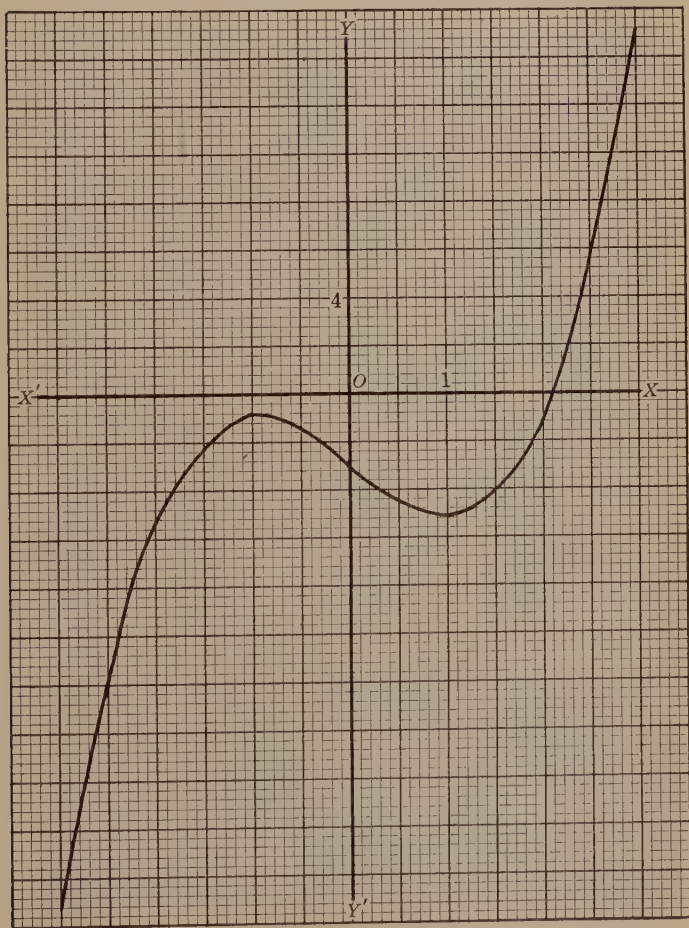
99. **Complex Roots Shown by a Graph.** — If a quadratic equation has two complex roots, its graph cannot cross or touch the x -axis; otherwise real roots would be shown.



Graph of $x^2 + x + 2$

Similarly in a cubic equation, if there are two complex roots the graph will cross the axis but once, as shown in the diagram. Hence *corresponding to every elbow of the curve*

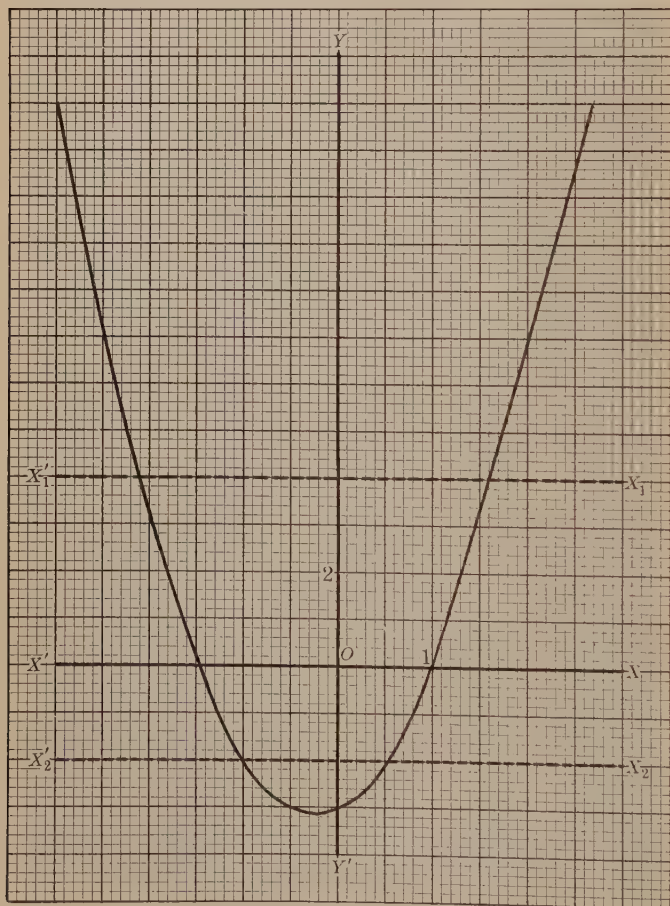
which does not intersect or touch the axis, there exists a pair of imaginary roots. Note that we cannot read from the graph what these roots are, but only that they exist. Is it possible for the graph of a cubic equation to cut the x -axis in two and two points only? Why?

Graph of $x^3 - 3x - 3$

100. Change of Axes. — Let it be required to solve by the same graph the equations

$$2x^2 + x - 3 = 0. \quad 2x^2 + x - 3 = 4. \quad 2x^2 + x - 3 = -2.$$

We obtain the roots of the first equation by taking the points of intersection of the graph of the function $2x^2 + x - 3$



Graph of $2x^2 + x - 3$

with the x -axis, whose equation is $y = 0$. Similarly we obtain the points where the function equals 4 by the points of intersection of the function graph with the line $y = 4$, and likewise solve the third equation by its intersection with the line $y = -2$. This is equivalent to changing the x -axis 4 units upward and 2 units downward respectively.

101. Derivative of a Function. When we have a rational integral function of x in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

a secondary function called the *first derived function* or the *first derivative* of this function may be found by operating upon each term by the following

Rule. — *Multiply the exponent of x in each term by the coefficient to form the new coefficient, and diminish the exponent by 1.*

The first derivative of $f(x)$ may be denoted by $f'(x)$.

Thus, if $f(x) = 2x^3 + 6x^2 - x + 3$,
then $f'(x) = 6x^2 + 12x - 1$.

Note that the derivative of 3 is zero.

The exact meaning of $f'(x)$ and its relation of $f(x)$ will be made clear when you study *calculus*. For the present, we shall use derived functions to obtain

102. Maxima and Minima Points of Graphs. — In order that we may draw an accurate graph of a function it is necessary that we should be able to find the *turning point* of the curve. A turning point is called a *maximum point* when the graph slopes downward from that point on both sides of it. It is called a *minimum point* when the graph slopes upward from that point on both sides of it. The way in which the derivative is used to locate the turning points of a graph may be made clear by the following

Rule. — (1) *Find the derivative of the function.*

(2) *Set the derivative equal to zero and solve the resulting equation.*

(3) *The roots, if real, give the abscissas of the turning points.*

(4) *Substitute them in the original function to obtain the ordinate of these points.*

Illustrative example :

Find the turning points of the graph of $4x^3 - 3x^2 - 6x + 2$.

Solution :

$$\text{If} \quad f(x) = 4x^3 - 3x^2 - 6x + 2,$$

$$f'(x) = 12x^2 - 6x - 6,$$

$$\text{then} \quad 12x^2 - 6x - 6 = 0.$$

$$2x^2 - x - 1 = 0.$$

$$(2x + 1)(x - 1) = 0.$$

$$x = 1, -\frac{1}{2}.$$

Substituting these values in $4x^3 - 3x^2 - 6x + 2$ to obtain corresponding values of $f(x)$, we obtain -3 and $3\frac{3}{4}$, respectively.

Hence the turning points are $(1, -3)$ and $(-\frac{1}{2}, 3\frac{3}{4})$.

If the derivative when set equal to zero has imaginary roots, the graph has no turning points.

Exercises

Find the first derivative of each of the following functions :

1. $x^2 + 3x - 6$.

4. $2x^4 + 6x^2 - 3$.

2. $2x^3 + 3x^2 - 2x$.

5. $3x^2 - 7x$.

3. $4x^3 - 2x^2 + 7x - 1$.

6. $8x^3 + 6x^2 - x + 7$.

7. Show that the turning point of illustrative graph on page 163 is $(\frac{1}{2}, -4\frac{1}{4})$.

8. Show that the turning points for illustrative graph on page 165 are $(\sqrt{3}, -6\sqrt{3} - 10)$ and $(-\sqrt{3}, 6\sqrt{3} - 10)$.

9. Show that the turning points for illustrative graph on page 166 are $(0, 5)$, $(\sqrt{3}, -4)$, and $(-\sqrt{3}, -4)$.

10. Show that the turning point for illustrative graph on page 168 is $(1, 0)$.

11. Show that the two turning points of illustrative graph on page 169 coincide in the point $(1, 0)$.

12. Show that the turning point of illustrative graph on page 170 is $(-\frac{1}{2}, 1\frac{3}{4})$.

13. Show that the turning points of illustrative graph on page 171 are $(1, -5)$ and $(-1, -1)$.

14. Show that the turning point of illustrative graph on page 172 is $(-\frac{1}{4}, -3\frac{1}{8})$.

Draw graphs of the following equations. Locate their turning points. Discuss the character of the roots as shown by the graph. Determine the roots, reading irrational roots to the nearest tenth.

$$15. 2x^2 + 5x - 3 = 0. \quad 18. -2x^2 - x + 6 = 0.$$

$$16. 2x^2 + 13x + 6 = 0. \quad 19. x^2 + 2x + 3 = 0.$$

$$17. 4x^2 - 12x + 9 = 0. \quad 20. -3x^2 + 4x - 10 = 0.$$

$$21. 4x^3 - 15x^2 + 18x + 6 = 0.$$

$$22. x^3 - 6x + 10 = 0. \quad 27. 2x^3 + 8x^2 + 11x = 0.$$

$$23. 3x^3 - 2x^2 + x - 2 = 0. \quad 28. 4x^4 - 37x^2 + 9 = 0.$$

$$24. x^3 - 1 = 0. \quad 29. x^4 - 2x^2 + 1 = 0.$$

$$25. x^3 - 9x + 3 = 0. \quad 30. x^4 + 2x^2 = 0.$$

$$26. 2x^3 - 3x^2 + 4 = 0. \quad 31. x^3 - x^2 - x + 1 = 0.$$

103. Fractional Roots. —

Theorem. — *An equation all of whose coefficients are integers, having 1 as the coefficient of the highest power, cannot have a rational fraction as a root.*

Proof: Given the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0,$$

where $p_1, p_2, \cdots p_n$ are integers.

Let us suppose that $\frac{a}{b}$, a rational fraction in its lowest terms, is a root of the equation.

$$\text{Then } \left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_n = 0.$$

Multiplying both members of the equation by b^{n-1} , we obtain

$$\frac{a^n}{b} + p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_nb^{n-1} = 0.$$

$$\text{Hence, } p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_nb^{n-1} = -\frac{a^n}{b}.$$

But the left member of this equation is integral, while the right member is a fraction in its lowest terms; for if a and b are prime to each other, so also are a^n and b . Since this is impossible $\frac{a}{b}$ cannot be a root of the equation.

104. Roots and Coefficients. — The equation whose roots are r_1, r_2, r_3 is

$$(x - r_1)(x - r_2)(x - r_3) = 0.$$

Multiplying out these factors, the equation becomes

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - (r_1r_2r_3) = 0.$$

Hence we see a very definite relation between the roots of the equation and the coefficients. The manner in which this result is obtained shows us that it is perfectly general, for if the roots are $r_1, r_2, r_3, \cdots r_n$, the corresponding equation is

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0,$$

which, when expanded, gives

$$x^n - (r_1 + r_2 + r_3 + \cdots + r_n)x^{n-1} + (r_1r_2 + r_1r_3 + r_2r_3 + r_1r_4 \cdots)x^{n-2} - (r_1r_2r_3 + r_1r_3r_4 + r_1r_3r_4 + \cdots)x^{n-3} + \cdots + (-1)^nr_1r_2r_3 \cdots r_n = 0.$$

The general equation of the n th degree in its simplest form is $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0$.

By comparing the coefficients we observe that

$$\begin{aligned} r_1 + r_2 + r_3 + \dots + r_n &= -p_1. \\ r_1r_2 + r_1r_3 + r_2r_3 + \dots &= p_2. \\ r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + \dots &= -p_3. \\ &\vdots \\ r_1r_2r_3 \dots r_n &= (-1)^n p_n. \end{aligned}$$

The relations here shown may be stated in the following

Principle. — *In a complete equation in standard form, the coefficient of whose highest power is unity,*

(1) *The sum of the roots is equal to the coefficient of the second term with its sign changed.*

(2) *The sum of the products of the roots taken two at a time is equal to the coefficient of the third term.*

(3) *The sum of the products of the roots taken three at a time is equal to the coefficient of the fourth term with its sign changed, etc.*

(4) *The product of all the roots is equal to the constant term, with the sign changed if the equation is odd degree.*

Illustrative example :

If $x^3 + 3x + 4 = 0$, the three roots r_1, r_2, r_3 , satisfy the following equations :

$$\begin{aligned} r_1 + r_2 + r_3 &= 0. \\ r_1r_2 + r_1r_3 + r_2r_3 &= 3. \\ r_1r_2r_3 &= -4. \end{aligned}$$

You should note :

(1) If the second term is missing, the sum of the roots is zero.

(2) If the absolute term is missing, at least one root is zero.

The above principle may be used to write an equation whose roots are given as shown in the following

Illustrative example :

Write the equation whose roots are $2 + \sqrt{3}$, $2 - \sqrt{3}$, 4 , -1 .

Solution: There are 4 roots; hence the first term is x^4 .

The sum of the roots is 7. Hence the coefficient of x^3 is -7 .
The sum of the products of the roots two at a time is

$$\begin{aligned} & (2 + \sqrt{3})(2 - \sqrt{3}) + 4(2 + \sqrt{3}) + 4(2 - \sqrt{3}) - 1(2 + \sqrt{3}) \\ & - 1(2 - \sqrt{3}) + 4(-1) = 1 + 8 + 4\sqrt{3} + 8 - 4\sqrt{3} - 2 \\ & - \sqrt{3} - 2 + \sqrt{3} - 4 = 9. \end{aligned}$$

Hence the coefficient of x^2 is $+9$.

Similarly we have

$$\begin{aligned} & (2 + \sqrt{3})(2 - \sqrt{3})4 + (2 + \sqrt{3})(2 - \sqrt{3})(-1) + 4(-1) \\ & (2 + \sqrt{3}) + 4(-1)(2 - \sqrt{3}) = 4 - 1 - 8 - 4\sqrt{3} - 8 + 4\sqrt{3} \\ & = -13. \end{aligned}$$

Hence the coefficient of x is $+13$.

$$-1(4)(2 + \sqrt{3})(2 - \sqrt{3}) = -4.$$

Since n is even, the constant term is -4 .

Therefore the equation is

$$x^4 - 7x^3 + 9x^2 + 13x - 4 = 0.$$

Notice that, if necessary, the number of products of any number of roots taken two at a time, three at a time, etc., can be found by the combination formulas of Chapter VI.

This principle also furnishes us a method of solving a numerical equation providing one other relation between the roots is known.

Illustrative example I:

Solve the equation $x^3 - 9x^2 + 11x - 21 = 0$, the roots of which are in arithmetic progression.

Solution: Denote the roots by $a - b$, a , $a + b$.

Then $3a = 9$, from which $a = 3$. (1)

$$a^2 - ab + a^2 + ab + a^2 - b^2 = 11. \quad (2)$$

$$3a^2 - b^2 = 11.$$

$$a^3 - ab^2 = 21. \quad (3)$$

$$27 - b^2 = 11. \quad \text{Sub. (1) in (3).}$$

$$b = \pm 4.$$

Hence the roots are $-1, 3, 7$.

Illustrative example II:

Solve the equation $4x^4 - 24x^3 + 55x^2 - 52x + 15 = 0$, given that the sum of two of the roots is 2.

Solution: Denote the roots by $a, 2 - a, b, c$.

$$\text{Then} \quad b + c + 2 = 6. \quad (1)$$

$$bc + ab + ac + 2b - ab + 2c - ac + 2a - a^2 = \frac{55}{4}. \quad (2)$$

$$abc + 2ab - a^2b + 2ac - a^2c + 2bc - abc = 13. \quad (3)$$

$$2abc - a^2bc = \frac{15}{4}. \quad (4)$$

$$b + c = 4. \quad (1)$$

$$bc + 2(b + c) + 2a - a^2 = \frac{55}{4}. \quad (2)$$

$$bc + 8 + 2a - a^2 = \frac{55}{4}.$$

$$bc + 2a - a^2 = \frac{23}{4}.$$

$$2a(b + c) - a^2(b + c) + 2bc = 13. \quad (3)$$

$$8a - 4a^2 + 2bc = 13$$

$$4a - 2a^2 + 2bc = \frac{23}{2}$$

$$\hline 2a^2 - 4a = -\frac{3}{2}$$

$$4a^2 - 8a + 3 = 0.$$

$$(2a - 1)(2a - 3) = 0.$$

$$a = \frac{1}{2}, \frac{3}{2}.$$

$$2 - a = \frac{3}{2}, \frac{1}{2}.$$

$$bc + \frac{3}{4} = \frac{23}{4}.$$

$$bc = 5.$$

$$b + c = 4.$$

Hence, $b = 2 + \sqrt{-1}$, or $2 - \sqrt{-1}$.
 $c = 2 - \sqrt{-1}$, or $2 + \sqrt{-1}$.

The roots, therefore, are $\frac{3}{2}$, $\frac{1}{2}$, $2 + i$, $2 - i$.

Exercises

1. State the sum and the product of the roots of

(a) $x^3 - 6x^2 - 4x + 3 = 0$. (c) $3x^4 - 2x^3 + 5x - 2 = 0$.

(b) $2x^3 + 5x^2 + 3x - 1 = 0$. (d) $2x^4 + 6x^2 - x - 2 = 0$.

2. State the sum of the products of the roots taken 3 at a time in (c) and (d) above.

3. State the number of products of roots taken 2 at a time and 3 at a time in (c) and (d) above.

Using the relation of roots to the coefficients, write the equations whose roots are

4. 2, 3, 0.

10. $2, \pm 3\sqrt{2}, \pm 2\sqrt{-1}$.

5. 3, -1, -1.

11. $3 \pm 2\sqrt{3}, -2 \pm 3\sqrt{-1}$.

6. $2, a \pm b$.

12. $8, 2, -3 \pm \sqrt{-7}$.

7. $3 \pm \sqrt{2}, 6$.

13. $\frac{3}{4}, 1 \pm \sqrt{3}, 1 \pm \sqrt{5}$.

8. $2 \pm \sqrt{-1}, -5$.

14. $\pm \sqrt{-1}, 3, \pm \sqrt{-2}, 2$.

9. 3, 0, 1; $\pm i$.

15. $-1 + \sqrt{-1}, -1 - \sqrt{-1}, 2$.

16. Using these relations as a means of checking, determine whether 2, 3, and -1 are roots of the equation $x^3 - 4x^2 - x + 6 = 0$.

17. Similarly determine whether $3, 2 \pm \sqrt{5}$, are roots of $x^3 - 7x^2 + 11x + 3 = 0$.

18. Solve $2x^3 - 27x^2 + 121x - 180 = 0$, given that the roots are in arithmetic progression.

19. Solve $2x^3 - 7x^2 + 7x - 2 = 0$, given that the roots are in geometric progression.

20. Solve $4x^4 - 16x^3 + 3x^2 + 4x - 1 = 0$, given that the sum of two of the roots is zero.

21. Solve $x^3 + 3x^2 - 10x - 24 = 0$, given that one root is twice another. $\triangleright$

22. Solve $x^4 + 2x^3 - 14x^2 - 30x + 9 = 0$, given that two roots are equal.

23. Solve $4x^3 - 24x^2 + 39x - 14 = 0$, given that the difference between two of the roots is 3.

24. Solve $6x^4 - 37x^3 + 136x^2 - 193x + 78 = 0$, given that the product of two of the roots is 1, and the sum of the other two is 4.

25. Given $x^4 - 6x^3 + mx^2 + nx + 24 = 0$; also that the sum of two of the roots is 4, and the product of the other roots is 6. Determine the values of m and n and solve the equation.

26. Solve $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of two of the roots being zero.

27. Solve $x^4 - 5x^3 - 5x^2 - 5x - 6 = 0$, if the sum of two of the roots is 5.

28. Solve $12x^3 - 43x^2 - 22x + 8 = 0$, if one root is the reciprocal of another.

29. Solve $x^4 - 10x^3 + 23x^2 + 10x - 24 = 0$, if the sum of two of the roots equals the sum of the other two and one of the roots is the negative of another.

30. Find k in $x^3 - 5x^2 - 4x + k = 0$, if the sum of two of the roots is zero.

105. Transformed Equations. — When an equation is so changed that each root of the equation is operated upon in the same way, the equation is said to be *transformed*.

For example, consider the equation

$$x^2 - x - 6 = 0.$$

If the roots of this equation are each multiplied by 2, the equation becomes

$$x^2 - 2x - 24 = 0.$$

If the signs of the roots are changed, the equation becomes

$$x^2 + x - 6 = 0.$$

If the roots are each diminished by 2, the equation becomes

$$x^2 + 3x - 4 = 0.$$

The way in which each of these transformations as well as others are performed will be made clear by the following propositions.

106. To Multiply the Roots of an Equation by a Constant. —

Given the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0.$$

Required to transform this equation into another one whose roots are equal to m times the roots of the given equation. Let the variable of the new equation be denoted by y .

Then, $y = mx$ or $x = \frac{y}{m}$.

Substituting in the given equation,

$$\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \dots + p_n = 0.$$

Clearing of fractions by multiplying each term by m^n ,

$$y^n + p_1my^{n-1} + p_2m^2y^{n-2} + \dots + p_nm^n = 0.$$

$$\text{Or } x^n + p_1mx^{n-1} + p_2m^2x^{n-2} + \dots + p_nm^n = 0.$$

Note that the required equation is obtained by multiplying the second term by m , the third by m^2 , etc.

Comparing this transformed equation with the given equation, we establish the following

Rule. — *To multiply the roots of an equation by a constant m ,*

(1) *Multiply the second highest power by m .*

(2) *Multiply the next highest power by m^2 and so on.*

(3) *Multiply the constant term by m^n .*

Example. — Multiply the roots of the equation

$$x^4 + 6x^3 + 3x - 1 = 0, \text{ by } 2.$$

Solution: We multiply the second term by 2, the third term (here missing) by 4, the fourth term by 8, the final term by 16, obtaining

$$x^4 + 12x^3 + 24x - 16 = 0.$$

Exercises

1. Form the equations which have

(a) Roots 3 times those of $x^3 - 6x^2 + 11x - 6 = 0$.

(b) Roots 2 times those of $x^3 + 6x + 7 = 0$.

2. Transform the equation $x^3 - 5x^2 - 7x + 11 = 0$ into an equation whose roots shall be 3 times the roots of the original equation.

3. Transform $x^3 - 5x^2 - 2x + 24 = 0$ into an equation whose roots shall be twice the roots of the original equation.

107. *To divide the roots of an equation by m , multiply them by $\frac{1}{m}$, using the above rule.*

108. *To change the signs of the roots of an equation, multiply them by -1 , by the rule of section 106.*

This is equivalent to *changing the sign of every alternate term beginning with the second.*

Note that the above statement assumes that the equation is complete and in standard form. If the equation is incomplete, that is, if some powers of x are missing, that fact must be taken into account in applying the rule, and the miss-

ing terms counted in determining which are the alternate terms. In such cases it may be simpler merely to *change the sign of every odd power of x* .

109. To Diminish the Roots of an Equation by a Constant.

— Given the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0,$$

required to transform the equation so as to diminish the roots by m .

Let y be the variable of the transformed equation.

Then, $y = x - m$, and $x = y + m$.

Substituting

$$(y + m)^n + p_1(y + m)^{n-1} + p_2(y + m)^{n-2} + \dots + p_n = 0.$$

Expanding and changing the y to x we have the transformed equation. We shall now illustrate by an

Example. — Diminish the roots of the equation

$$x^3 + 2x^2 + 7x - 3 = 0 \text{ by } 2.$$

Solution: The equation is

$$(y + 2)^3 + 2(y + 2)^2 + 7(y + 2) - 3 = 0.$$

$$\text{Or, } y^3 + 6y^2 + 12y + 8 + 2y^2 + 8y + 8 + 7y + 14 - 3 = 0.$$

$$\text{Or, } y^3 + 8y^2 + 27y + 27 = 0.$$

$$\text{Or, } x^3 + 8x^2 + 27x + 27 = 0.$$

We shall now explain a shorter process of obtaining the above transformation.

Since $y = x - 2$, the transformed equation

$$y^3 + 8y^2 + 27y + 27 = 0, \text{ is the same thing as}$$

$$(x - 2)^3 + 8(x - 2)^2 + 27(x - 2) + 27 = 0.$$

We can get the coefficients 27, 27, 8 in succession by dividing this equation by $(x - 2)$ and taking the remainder, again dividing by $(x - 2)$ and taking the remainder, etc., thus obtaining the coefficients in reverse order.

Also the equation $y^3 + 8y^2 + 27y + 27 = 0$, which is the same as the above equation, differs from the original equation only in having the variable $y + 2$ instead of x . Now the remainder obtained by dividing by a quantity of the form $x - a$ is numerical, and hence is independent of the variable used. It follows then that these same coefficients may be obtained by dividing the given equation successively by $x - 2$.

Hence by synthetic division,

$$\begin{array}{r|rrrr}
 1 & 2 & 7 & -3 & 2 \\
 & 2 & 8 & 30 & \\
 \hline
 1 & 4 & 15 & 27 & \\
 & 2 & 12 & & \\
 \hline
 1 & 6 & 27 & & \\
 & 2 & & & \\
 \hline
 1 & 8 & & &
 \end{array}$$

Taking the successive remainders as coefficients of the transformed equation, we obtain as before

$$x^3 + 8x^2 + 27x + 27 = 0.$$

This method of *transforming* the equation must not be confused with simple *division*.

For example, the quotient obtained by dividing

$$x^3 + 2x^2 + 7x - 3 \text{ by } (x - 2) \text{ is } x^2 + 4x + 15$$

with a remainder of 27.

Increasing the roots of an equation by a given number may be effected by the same process when we recall that increasing the roots by m is the same as diminishing them by $-m$.

Exercises

1. Diminish the roots of $2x^3 - 7x^2 - 3x + 1 = 0$ by 4.
2. Transform the equation $x^4 - 12x^3 + 17x^2 - 9x + 7 = 0$ into another whose roots shall be less by 3.

3. Transform the equation $x^3 - 5x^2 - 2x + 24 = 0$ into another whose roots shall be diminished by 2.

4. Given the equation $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$, write the equation whose roots are each 2 more than the roots of the given equation.

5. Given the equation $3x^3 - x^2 - 5 = 0$, write the equation whose roots are each 1 less than the roots of the given equation.

110. Removal of Second Term of an Equation. — For some purposes an equation is simpler when the second highest power of the unknown is lacking. This may always be effected by an application of the preceding transformation as shown by the following

Illustrative example :

By a transformation, remove the coefficient of x^2 from the equation $x^3 - 6x^2 + 3x - 2 = 0$.

Solution : The sum of the roots is 6. In order that the sum may be made zero, we must diminish the present sum by 6.

Hence, since there are three roots, we must diminish each root by 2.

$$\begin{array}{r|rrrr}
 1 & -6 & 3 & -2 & 2 \\
 & 2 & -8 & -10 & \\
 \hline
 1 & -4 & -5 & -12 & \\
 & 2 & -4 & & \\
 \hline
 1 & -2 & -9 & & \\
 & 2 & & & \\
 \hline
 1 & 0 & & &
 \end{array}$$

The transformed equation is $x^3 - 9x - 12 = 0$.

In general, to remove the second term from the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$, we must diminish the roots by $-\frac{p_1}{n}$.

Exercises

1. Transform the equation $x^3 - 6x^2 + 4x + 5 = 0$ into another lacking the term in x^2 .

2. Transform the equation $x^3 + 6x^2 + 8x - 2 = 0$ into an equation in which the x^2 term shall be missing.

3. Transform the equation $x^3 + 9x^2 + 12x + 19 = 0$ into another in which the term in x^2 shall be missing.

4. Transform $2x^3 - 9x^2 + 4x + 6 = 0$ into an equation wanting the second term.

111. To Change the Roots of an Equation into Their Reciprocals. — Given the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0,$$

required to write an equation whose roots are the reciprocals of the roots of the given equation.

Let $y = \frac{1}{x}$, then $x = \frac{1}{y}$.

Substituting,

$$\frac{1}{y^n} + \frac{p_1}{y^{n-1}} + \frac{p_2}{y^{n-2}} + \cdots + p_n = 0.$$

Clearing of fractions,

$$1 + p_1y + p_2y^2 + \cdots + p_ny^n = 0.$$

Or, $p_nx^n + p_{n-1}x^{n-1} + \cdots + p_2x^2 + p_1x + 1 = 0$.

Comparing this equation with the given equation, we note that the coefficients occur in reverse order.

Hence, *to change the roots of an equation into their reciprocals, reverse the order of the coefficients.*

Illustrative example :

Write the equation whose roots are the reciprocals of the roots of $2x^3 - 3x^2 - 4x + 5 = 0$.

Solution: By applying the above principle we obtain

$$5x^3 - 4x^2 - 3x + 2 = 0.$$

Exercises

Write the equation which shall have roots which are the reciprocals of each of the following equations.

1. $x^3 - 7x^2 - 4x + 2 = 0$. 4. $2x^4 + 2x^2 - 6x + 3 = 0$.

2. $x^4 - 8x^2 + 7 = 0$. 5. $6x^3 - 3x^2 + x - 5 = 0$.

3. $x^5 - x^3 + 5x^2 + 8 = 0$.

112. Illustrative Use of Transformations. —

1. *To transform an equation with fractional coefficients into another whose coefficients shall be integral, that of the first term being unity.*

This is accomplished by multiplying the roots of the equation by m (see section 106), and then by giving m the least value as will make all the coefficients integral.

Example. — Transform $x^3 - \frac{x^2}{3} - \frac{x}{36} + \frac{1}{108} = 0$ into an equation with integral coefficients, and that of the first term being unity.

Solution: Multiplying the roots by m , we obtain

$$x^3 - \frac{mx^2}{3} - \frac{m^2x}{36} + \frac{m^3}{108} = 0.$$

Now note that the least value of m which will make all the coefficients integral is 6.

Now by letting $m = 6$, we obtain

$$x^3 - 2x^2 - x + 2 = 0,$$

whose roots are 6 times those of the given equation.

2. *To transform the equation $36x^3 + 18x^2 + 2x + 9 = 0$ into an equation which has integral coefficients and unity for the coefficient of x^3 .*

Solution: Dividing by 36, we obtain

$$x^3 + \frac{x^2}{2} + \frac{x}{18} + \frac{1}{4} = 0.$$

Multiplying the roots by m (see section 106),

$$x^3 + \frac{mx^2}{2} + \frac{m^2x}{18} + \frac{m^3}{4} = 0.$$

We now note that the least (or smallest) value of m , which will cause all of the denominators to disappear, will be 6.

$\therefore$ We now substitute 6 for m in the above equation, and obtain

$$x^3 + 3x^2 + 2x + 54 = 0,$$

which has the required form. The roots of this equation, each divided by 6, are the roots of the given equation.

Exercises

Transform each of the following into an equation with integral coefficients, and that of the first term being unity.

$$1. \quad x^3 + 2x^2 + \frac{x}{4} - \frac{1}{8} = 0. \quad 3. \quad x^3 + \frac{3x^2}{2} + \frac{x}{8} - 1 = 0.$$

$$2. \quad x^3 + \frac{x^2}{3} - \frac{x}{9} + 2 = 0. \quad 4. \quad x^3 - \frac{2x}{5} + \frac{13}{20} = 0.$$

$$5. \quad x^3 + \frac{13}{6}x^2 + \frac{x}{6} - \frac{1}{3} = 0.$$

$$6. \quad x^4 - \frac{9}{2}x^2 + 8x + \frac{33}{16} = 0.$$

$$7. \quad x^4 - 3x^3 - \frac{5}{2}x^2 + \frac{5}{4}x + \frac{1}{16} = 0.$$

$$8. \quad x^4 - \frac{7x^2}{45} - \frac{2x}{75} + \frac{2}{375} = 0.$$

$$9. \quad x^4 + \frac{3x^2}{10} + \frac{13x}{25} + \frac{77}{1000} = 0.$$

$$10. \quad 10x^4 + 20x^3 - 4x^2 + 25x - 30 = 0.$$

11. $2x^4 - 3x^3 + 5x^2 - 4x + 6 = 0.$

12. $108x^3 - 36x^2 - 6x + 1 = 0.$

13. $4x^4 - 8x^3 - 6x^2 + 4x - 1 = 0.$

14. $3x^5 + x^4 - 5x^3 + 2x^2 - 7x + 5 = 0.$

15. $2x^4 - x^3 - 29x^2 + 34x + 24 = 0.$

16. What fact do we know about the roots of each of these transformed equations?

See section 103.

113. Other Transformations. — (1) *To write an equation whose roots are the squares of the roots of a given equation.*

Let $y = x^2$. Then $x = \sqrt{y}$.

Make the substitutions and rationalize the equation. The details are left as an exercise for the student.

(2) *To write an equation whose roots are the square roots of the roots of a given equation.*

Let $y = \sqrt{x}$. Then $y^2 = x$. Substitute.

The details are left as an exercise for the student. Why is the degree of the equation, and hence the number of roots, changed in this case?

Exercises

Transform each of the following equations so as to change the roots to their respective reciprocals.

1. $x^4 + 6x^3 - x^2 + 3x + 1 = 0.$

2. $x^3 - 6x^2 + 3 = 0.$

3. $4x^3 + 6x - 2 = 0.$

4. $2x^4 - 2x^2 - 7x - 1 = 0.$

5. Multiply the roots of the equation

$$2x^4 + 3x^3 - x^2 + x - 1 = 0$$

by 2. By 3. By -2 .

6. Change the signs of the roots of

(a) $6x^3 + x^2 - 3 = 0$. (b) $x^4 + 6x^2 + 3x - 1 = 0$.

7. Divide the roots of the equation

$$x^3 + 2x^2 - 8x + 7 = 0$$

by 2. By 3. By -5 .

8. Diminish the roots of the equation

$$2x^4 - 3x^3 + 8x - 6 = 0$$

by 2. By 3. By 1. By $\frac{1}{2}$.

9. Increase the roots of the equation

$$x^3 + 5x^2 - 8x + 1 = 0$$

by 2. By 3. By 6.

10. Write the equation whose roots are the squares of the roots of

(a) $x^3 + 6x^2 - x + 2 = 0$. (b) $x^4 + 6x^2 - 3 = 0$.

11. Write the equation whose roots are the cubes of the roots of $x^3 + 6x - 2 = 0$.

12. Write the equation whose roots are the square roots of the roots of $2x^3 - 3x^2 + x - 5 = 0$.

By a transformation, remove fractional roots from each of the following equations and state what you have done to the roots in each case.

13. $2x^4 + 7x^3 - 3x^2 + x - 1 = 0$.

14. $3x^5 - 2x^3 - 4x^2 + x - 3 = 0$.

15. $x^3 + \frac{2}{3}x^2 - \frac{1}{6}x + 2 = 0$.

16. $\frac{1}{2}x^4 + \frac{2}{3}x^3 - 6x^2 + 1 = 0$.

17. $4x^4 + x^3 - \frac{1}{2}x^2 + x - 2 = 0$.

By a transformation remove the second term from each of the following equations, and state what you have done to the roots in each case.

$$18. x^3 + 3x^2 - 5x + 2 = 0.$$

$$19. x^3 - 6x^2 - 3x + 7 = 0.$$

$$20. x^4 - 12x^3 - 2x^2 + 9x - 2 = 0.$$

$$21. 3x^4 + 12x^3 + 6x - 2 = 0.$$

22. By a transformation, remove the coefficient of x from the equation

$$x^3 + 11x^2 + 40x + 16 = 0.$$

Hint: Diminish the roots by the constant k . Then determine the value of k so that the coefficient of x shall be zero.

114. Graphical Meaning of Transformations. — Suppose the equation

$$x^2 - 2x - 8 = 0$$

is transformed

(a) by diminishing the roots by 2.

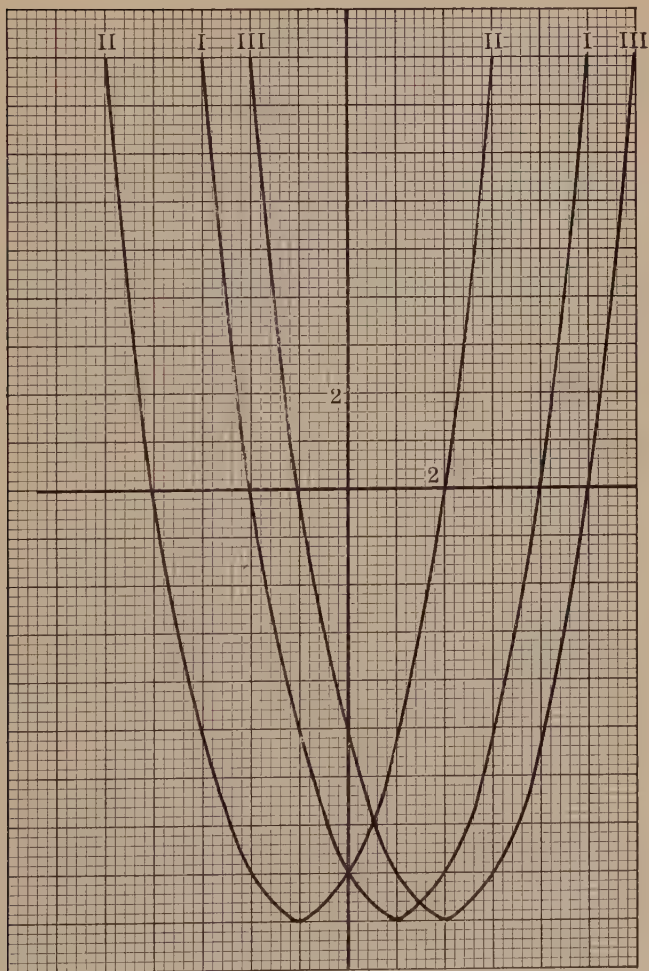
(b) by increasing them by 1.

The resulting equations are $x^2 + 2x - 8 = 0$ and $x^2 - 4x - 5 = 0$, respectively.

We shall graph all of these equations on the same axes, numbering the graphs I, II, and III, respectively (page 193).

These graphs are identical in *form*, differing only in *position*. In graph II, the entire figure is shifted two units to the left, and in graph III, the entire figure is shifted one unit to the right. Thus we see clearly from the graph that both roots are diminished by 2 in the one case and increased by 1 in the other case.

115. Trial Solution of Equations with Rational Roots. — Numerical equations, some or all of whose roots are rational, are usually solved most easily by trial, using synthetic division.



Graphs of $x^2 - 2x - 8$
 $x^2 + 2x - 8$
 $x^2 - 4x - 5$

When the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

is written in the form

$$x_n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_n}{a_0} = 0,$$

it was shown in section 104 that p_n or $\frac{a_n}{a_0}$ is numerically the

product of the roots. It follows that every root of the equation is a factor of this term. Hence *every integral root of the equation is a factor of a_n* , and if the equation has a root, which is a rational fraction, *its numerator is a factor of a_n and its denominator a factor of a_0* .

Illustrative example I:

Solve the equation $x^3 - 5x^2 + 11x - 10 = 0$.

Solution: If a is a root of this equation, $x - a$ is a factor of its left member. Hence when it is divided by $x - a$, the remainder must be 0. The equation cannot have a rational fractional root (section 103). Any integral root must be a factor of 10. The possibilities are $\pm 1, \pm 2, \pm 5, \pm 10$.

$$\begin{array}{r} \text{Trial using 1.} \quad 1 \quad -5 \quad 11 \quad -10 \quad | \quad 1 \\ 1 \quad -4 7 \\ \hline 1 \quad -4 7 \quad | \quad -3 \end{array}$$

Since the remainder is not 0, 1 is not a root. Similarly -1 is not a root.

$$\begin{array}{r} \text{Trial using 2.} \quad 1 \quad -5 \quad 11 \quad -10 \quad | \quad 2 \\ 2 \quad -6 10 \\ \hline 1 \quad -3 5 \quad | \quad 0 \end{array}$$

Since the remainder is 0, 2 is a root. The quotient or depressed equation is

$$x^2 - 3x + 5 = 0.$$

Then
$$x = \frac{3 \pm \sqrt{9 - 20}}{2}.$$

Hence the roots are $2, \frac{3 \pm \sqrt{-11}}{2}.$

Illustrative example II:

Solve the equation $2x^3 - x^2 - 5x + 3 = 0.$

Solution: Any integral root must be a factor of 3. If there is a fractional root, its numerator must be a factor of 3 and its denominator a factor of 2. Possibilities are $\pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}.$

If we try each of these numbers in succession, we find no root until we come to $\frac{3}{2}.$

$$\begin{array}{r} 2 \quad -1 \quad -5 \quad 3 \mid \frac{3}{2} \\ \quad 3 \quad 3 \quad -3 \\ \hline 2 \quad 2 \quad -2 \quad 0 \end{array}$$

Hence $\frac{3}{2}$ is a root.

$$2x^2 + 2x - 2 = 0.$$

Or $x^2 + x - 1 = 0.$

Then
$$x = \frac{-1 \pm \sqrt{1 + 4}}{2}.$$

Hence the roots are $\frac{3}{2}, \frac{-1 \pm \sqrt{5}}{2}.$

Sometimes it is best to make use of a transformation before attempting to solve as in

Illustrative example III:

Solve the equation $4x^4 - 4x^3 - x^2 - 5x + 3 = 0.$

Solution: In order that we may not have to work with fractional roots we shall transform the equation so that all rational roots are integral. (Section 103.)

Dividing this equation by 4, we obtain

$$x^4 - x^3 - \frac{1}{4}x^2 - \frac{5}{4}x + \frac{3}{4} = 0.$$

Multiplying roots by 2, the smallest number that will remove fractional coefficients, we obtain

$$x^4 - 2x^3 - x^2 - 10x + 12 = 0.$$

We now solve the transformed equation as follows

$$\begin{array}{rrrrr} 1 & -2 & -1 & -10 & 12 & \overline{)1} \\ & 1 & -1 & -2 & -12 & \\ \hline 1 & -1 & -1 & -12 & & 0 \end{array}$$

Hence 1 is a root.

Using the depressed equation,

$$\begin{array}{rrrr} 1 & -1 & -2 & -12 & \overline{)3} \\ & 3 & 6 & 12 & \\ \hline 1 & 2 & 4 & & 0 \end{array}$$

Hence 3 is a root.

Solving, $x^2 + 2x + 4 = 0$.

$$x = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm \sqrt{-3}.$$

Hence the roots of the transformed equation are 1, 3, $-1 \pm \sqrt{-3}$.

But the roots of the transformed equation are twice as great as the roots of the original equation. Therefore the

roots of the original equation are $\frac{1}{2}, \frac{3}{2}, -\frac{1 \pm \sqrt{-3}}{2}$.

Exercises

Solve each of the following equations:

1. $x^3 - 5x^2 - 9x + 45 = 0.$ —

2. $2x^3 - 5x^2 - 9x + 18 = 0.$

3. $6x^3 + 11x^2 - x - 6 = 0$.
4. $2x^3 - x^2 - 7x - 3 = 0$.
5. $4x^3 - 4x^2 - 7x + 6 = 0$.
6. $3x^3 - 5x^2 + 3x - 5 = 0$.
7. $x^4 - x^3 - 7x^2 + x + 6 = 0$.
8. $x^4 - 5x^3 - 4x^2 + 17x + 15 = 0$.
9. $2x^4 - 11x^3 + 21x^2 - 16x + 4 = 0$.
10. $3x^4 - x^3 - 7x^2 + 4x - 20 = 0$.
11. $2x^4 - x^3 - 15x^2 + 9x - 27 = 0$.
12. $6x^4 - 13x^3 - 26x^2 + 38x + 15 = 0$.
13. $8x^4 - 2x^3 - 41x^2 - 34x - 6 = 0$.

Review

1. Simplify $(i^9 + i^{10} + i^{11} + i^{12})^7$, if $i = \sqrt{-1}$.
2. How many different numbers of seven figures can be formed from the digits 1, 1, 2, 3, 3, 3, 4?
3. One root of the equation $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$ is $2 - \sqrt{3}$. Find the other roots.
4. State the expressions for the coefficients of an equation of degree n in terms of the roots.
5. Form an equation with integral coefficients two of whose four roots are $2 - \sqrt{-3}$ and $\sqrt{5}$.

116. Descartes' Rules of Signs. — The celebrated French mathematician René Descartes laid down rules for determining the upper limit of the number of positive and negative real roots of an equation, which may be stated as follows:

(1) *The number of real positive roots of the equation $f(x) = 0$ cannot exceed the number of variations of sign in $f(x)$.*

(2) *The number of real negative roots of the equation $f(x) = 0$ cannot exceed the number of variations in $f(-x)$.*

We shall now explain the meaning of the terms used in the above rule. When the equation $f(x) = 0$ is arranged in standard form, two consecutive signs which are unlike constitute a *variation* of sign. Thus the equation

$$2x^4 - 3x^3 - 6x + 2 = 0$$

has two variations of sign. The first term $+$, the second $-$, constitute a variation; the third term $-$, the fourth term $+$, constitute a second variation. Hence by the first rule stated above this equation cannot have more than two positive real roots.

$f(-x)$ means the result obtained by substituting $-x$ for x in the above expression.

In this case $f(-x) = 2x^4 + 3x^3 + 6x + 2$.

Since this has no variation of sign, the equation can have *no* negative root, by the second rule. It follows that the equation has at least two complex roots because it has four roots two of which at most are real.

You will notice that changing the x to $-x$ in the given equation is equivalent to changing the signs of the roots. Hence we really test for the negative roots of the given equation by means of the positive roots of the transformed equation.

In seeking the reason for these rules we may observe that an equation of the first degree with one positive root must have exactly one variation of sign. Ex. $x - 3 = 0$.

If each member of this equation is multiplied by the factor $x - 2$, a second positive root is introduced. The equation now becomes $x^2 - 5x + 6 = 0$, in which there are two variations of sign. If we again multiply by $x - 1$, thus introducing a third positive root, the equation becomes

$x^3 - 6x^2 + 11x - 6 = 0$, in which there are three variations. Now let us write the signs only of the last polynomial of the equation

$$+ - + -$$

In this there are three changes of sign.

Now we wish to show that if this polynomial be multiplied by a binomial whose signs, corresponding to a positive root, are $+ -$, the resulting polynomial will have at least one more change of sign than the original.

Writing down only the signs that occur in the operation, we have

$$\begin{array}{r}
 + - + - \\
 + - \\
 \hline
 + - + - \\
 \quad - + - + \\
 \hline
 + - + - +
 \end{array}$$

We now observe that there are four changes of sign. If this were attempted with a larger polynomial, it might be shown that while in certain cases terms of doubtful sign will occur, still one variation of sign will be added.

In general it can be proved that the multiplication of any such function by a factor whose signs are $+ -$, corresponding to a positive root, introduces *at least one* additional variation of sign into the function, so that the number of positive roots cannot exceed the number of variations.

You should *avoid the mistake* of supposing that the number of positive roots is necessarily the same as the number of variations of sign, for the number of variations may exceed the number of positive roots.

For example, the equation $x^2 - x + 1 = 0$ has two imaginary roots, $\frac{1 \pm \sqrt{-3}}{2}$, though there are two variations of sign.

117. Sign of the Constant Term. — We have seen in the preceding section that Descartes' rules of signs merely give us an upper limit to the number of positive and negative real roots of an equation. Additional information can often be obtained by considering the sign of the constant term.

By section (104) the sign of the constant term is the sign of the product of the roots if n is even, and the reverse if n is odd. Hence we know the sign of the product of the roots in any case. Since the product of conjugate imaginaries is positive, and the product of an even number of negatives is positive, it follows that an equation must have an *odd number* of negative real roots if the sign of the product of its roots is negative.

Illustrative example :

Obtain all the information possible about the roots of the equation $x^8 - 1 = 0$.

Solution: There is one variation of sign in $f(x)$, hence there is *not more than 1* positive root.

$f(-x)$ is identical with $f(x)$, hence there is *not more than 1* negative root.

But the product of the roots is negative, hence there is an odd number of negative roots and this number must be 1. Also since there is an even number of imaginaries there must be 1 real positive root also. We conclude that the equation has 1 positive, 1 negative, and 6 complex roots.

The following additional truths may be inferred at once from Descartes' rules.

(1) *If the signs of all the terms of an equation are positive, the equation has no real positive root.*

(2) *If the signs of the terms of a complete equation are alternately positive and negative, the equation has no real negative roots.*

(3) *If the signs of all terms of an equation are positive and only even powers are present, the roots of the equation are all complex, for $f(-x)$ is identical with $f(x)$ and there are no variations of sign in either.*

Illustrative example I :

Determine the nature of the roots of

$$x^3 - 6x^2 - 8x - 3 = 0.$$

Solution : There is 1 variation of sign, hence not more than 1 positive root.

$f(-x) = -x^3 - 6x^2 + 8x - 3$. There are 2 variations, hence not more than 2 negative roots.

As the sign of the product of the roots is positive, there may be 1 positive and 2 negatives, or 1 positive, 0 negatives, and 2 complex roots.

Which of these two possibilities is actually the case may be determined by graphing the function.

Illustrative example II :

Determine the nature of the roots of

$$x^4 + x^2 - 6 = 0.$$

Solution : There is 1 variation of sign, hence not more than 1 positive root.

$f(-x)$ is identical with $f(x)$. Hence there is not more than 1 negative root.

Since there are 4 roots in all, there are at least 2 complex roots.

The product of the roots is negative, hence there must be 1 negative root.

Since the number of complex roots is even, there is 1 positive root.

Hence the equation has 1 positive, 1 negative, and 2 complex roots.

Exercises

Determine as far as possible the nature of the roots of :

1. $2x^3 - x^2 - x - 6 = 0$.
2. $x^3 + x^2 + 3 = 0$.
3. $x^4 - x^3 - x + 2 = 0$.
4. $x^4 - x^3 + 6x^2 + 3x - 2 = 0$.
5. $x^3 + x^2 - 2 = 0$.
13. $x^3 + x - 6 = 0$.
6. $2x^3 + x + 1 = 0$.
14. $x^7 + x^4 - x - 1 = 0$.
7. $x^3 - 2 = 0$.
15. $x^8 + x^2 + x + 1 = 0$.
8. $2x^4 + x^2 + 1 = 0$.
16. $x^n - 1 = 0$. (n , odd.)
9. $x^6 + x^3 - 8 = 0$.
17. $x^n + 1 = 0$. (n , odd.)
10. $x^6 - x^4 + x^2 - 1 = 0$.
18. $x^{2n} - 1 = 0$.
11. $x^6 - 1 = 0$.
19. $x^{2n} + 1 = 0$.
12. $x^5 + 16 = 0$.
20. $x^{4n} + x^{2n} + 1 = 0$.
21. $x^6 - 4x^3 + x^2 - 11 = 0$.
22. $x^6 - 5x^4 + 3x - 7 = 0$.
23. $x^6 - 2x^3 + x^2 - 11 = 0$.
24. From a study of the signs of

$$x^3 + ax^2 + a^2 = 0,$$

say what you can about the roots of the equation under the two hypotheses (1) that a is positive and (2) that a is negative.

25. $x^5 + 2x^2 - x - 11 = 0$.
26. $x^5 + 3x^3 - 2x^2 - x + 11 = 0$.
27. $x^5 + 2x^3 - 5x^2 + x + 11 = 0$.
28. Determine the character of the roots of the equation

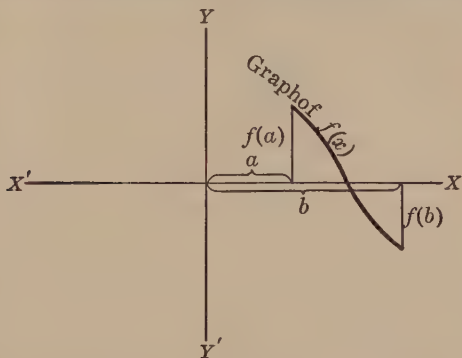
$$x^3 + bx + c = 0$$

(1) when b and c are both positive, (2) when b is negative and c positive.

118. Location of Roots. — A real root of an equation may be located between any given limits by means of the following

Principle: *In considering the equation $f(x) = 0$, when for two values of x , say a and b , $f(a)$ and $f(b)$ have opposite signs, a real root of the equation lies between a and b .*

This may be clearly shown by considering the graph of $f(x)$.



Let us suppose that $f(a)$ is positive and $f(b)$ negative as shown in the diagram. It may be proved that every algebraic function is a *continuous function*. This means that as x changes gradually from a to b , passing through every intervening value, $f(x)$ will change gradually to $f(b)$, passing through every intervening value. Hence the function in passing from a positive value to a negative value, or vice versa, must pass through zero.

It is plain that there may be more than one real root between a and b if $f(a)$ and $f(b)$ have opposite signs, but if so, there must be an odd number of them. If a and b are taken near together, it is usually safe to assume that there is but one.

Similarly if $f(a)$ and $f(b)$ have the same signs, there is either no root of the equation between a and b , or an even number of roots. Here again if a and b are near together, we may usually assume that there is no real root between a and b .

This principle is usually used to locate a real root of an equation between consecutive integers.

Illustrative example :

Locate the real roots of the equation

$$x^4 + 2x^2 - 6 = 0.$$

By substituting successive integers for x we obtain the following table :

$$f(0) = -6.$$

$$f(1) = -3.$$

$$f(2) = 18.$$

$$f(-1) = -3.$$

$$f(-2) = 18.$$

Since $f(1)$ and $f(2)$ have opposite signs, a real root of the equation lies between 1 and 2. Also since $f(-1)$ and $f(-2)$ have opposite signs, a real root lies between -1 and -2 . By Descartes' Rules the equation has only two real roots, so that we know that we have located all the real roots.

Exercises

Determine the location of the real roots in each of the following equations :

1. $x^3 + 6x^2 - 3 = 0.$

2. $3x^3 + 5x^2 + 2x + 1 = 0.$

3. $x^4 - 2x^2 + 3 = 0.$

4. $x^4 - 2x^3 + 3 = 0.$

5. $x^3 + 2x^2 - 4x - 5 = 0.$

6. $x^4 - 3x^2 - x + 2 = 0.$

7. $x^5 - 40 = 0.$

8. $2x^4 + 3x^3 - x - 8 = 0.$

9. $x^3 + 3x^2 - 4x - 8 = 0.$

10. $2x^3 - 7x^2 + x + 2 = 0.$

$$11. x^3 + 2x^2 - 13x + 11 = 0.$$

$$12. x^3 - 4x^2 - 4x + 12 = 0.$$

$$13. 3x^4 + 4x^3 - 36x^2 - 5 = 0.$$

$$14. x^3 - 2x^2 - 7x - 1 = 0.$$

119. Horner's Method for Irrational Roots. — We have seen how to determine the character of the roots of a numerical equation, how to find those that are rational, and how to locate irrational roots between consecutive integers and to estimate them to the nearest tenth by means of a graph. We shall now show how irrational roots may be found in decimal form to any required degree of accuracy.

You should study very carefully the solution of the following

Illustrative example I:

Find to the nearest hundredth a positive root of

$$x^3 - 6x^2 + 5x - 3 = 0.$$

Solution:

$$f(0) = -3.$$

$$f(1) = -3. \quad \text{Hence, we locate the root between 5 and 6.}$$

$$f(2) = -9. \quad \text{We shall now diminish the roots of the}$$

$$f(3) = -15. \quad \text{equation by 5.}$$

$$f(4) = -15.$$

$$f(5) = -3. \quad \begin{array}{rrrr} 1 & -6 & 5 & -3 \end{array} \bigg| 5$$

$$f(6) = 27. \quad \begin{array}{rrrr} & 5 & -5 & 0 \\ \hline 1 & -1 & 0 & -3 \end{array}$$

$$\begin{array}{rr} & 5 & 20 \\ \hline 1 & 4 & 20 \end{array}$$

$$\begin{array}{rr} & 5 \\ \hline 1 & 9 \end{array}$$

$$\begin{array}{rr} 1 & 9 \end{array}$$

The transformed equation is $x^3 + 9x^2 + 20x - 3 = 0$. This equation has a root between 0 and 1. Its approxi-

Here again the fact that the constant term does not change sign shows that .04 is not too large.

We shall now diminish the roots of this equation by .005 to determine whether the root of the original equation is greater or less than 5.145.

1	9.42	22.5788	— .020856	.005
	.005	.047125	.113129625	
1	9.425	22.625925	+	

Since this transformation changes the sign of the constant term, we know that .005 is too large. Hence the root of the given equation lies between 5.14 and 5.145 and therefore is 5.14 to the *nearest hundredth*.

Having completed the solution of this example we shall now explain the reasons for some of the steps which may not be clear to the pupil.

We shall first explain why dividing the constant term of a transformed equation by the coefficient of x gives approximately the next figure of the root. The root of the transformed equation must be less than 1. When the value of x is very small, the value of the higher powers of x become negligible as compared with the value of the term involving x , so that such an equation as

$$x^3 + 9x^2 + 20x - 3 = 0$$

becomes approximately $20x - 3 = 0$, and the root of this equation, of course, found by dividing 3 by 20.

However, these neglected terms do have some value, so that the result obtained by the above process is not always accurate. It is frequently too large, but rarely, if ever, too small. If it is too large, that fact will be discovered when we come to diminish by it. Obviously the process will give more and more accurate results as the solution proceeds and the value of x becomes smaller.

The reason why the constant term changes sign when we diminish by a number which is too large, becomes clear, when we remember how the constant term is related to the product of the roots. If we diminish the roots of the equation by a number which is larger than the root we are seeking, we thereby change the sign of that root, and hence change the sign of the product of the roots.

We are now prepared to state the following

Rule. —

(1) *By use of Descartes' rules, inspection of the constant term, and if necessary by a graph, determine the number of real roots of the equation.*

(2) *Locate the roots between consecutive integers.*

Then find each positive root as follows :

(1) *Diminish the roots of the equation by the smaller of the integers between which the root lies.*

(2) *Determine approximately the second figure of the root by dividing the constant term by the coefficient of x in the transformed equation.*

(3) *Diminish the roots of the transformed equation by the approximate second figure of the root.*

If this transformation causes the constant term to change sign, it shows that the root is too large, and it must be diminished until the sign of the constant term is not changed.

(4) *Obtain the third figure of the root by dividing the constant term by the coefficient of x , diminish by the result, and so continue until the desired number of places have been obtained.*

(5) *Adjust the decimal place of the answer to the nearest tenth or the nearest hundredth as may be required.*

To obtain a real negative root of an equation it is best to transform the equation so as to change the signs of the roots.

The negative root is thus made positive and may be found by the above rule. After it is found, place a *minus* sign in front of it.

Illustrative example II:

Find to three decimal places, a real negative root of the equation

$$2x^3 + 6x^2 - x + 4 = 0.$$

Solution:

$f(0) = +4$. The real root lies between -3 and -4 .

$f(-1) = +9$. We now change the signs of the roots thus,

$$f(-2) = +14. \quad 2x^3 - 6x^2 - x - 4 = 0.$$

$f(-3) = +7$. The corresponding roots of this equation

$f(-4) = -24$. lie between 3 and 4.

$$\begin{array}{r} 2 \quad -6 \quad -1 \quad -4 \quad \underline{3} \\ \quad 6 \quad 0 \quad -3 \end{array}$$

$$\begin{array}{r} 2 \quad 0 \quad -1 \quad \quad -7 \\ \quad 6 \quad 18 \end{array}$$

$$\begin{array}{r} 2 \quad 6 \quad \quad 17 \\ \quad 6 \end{array}$$

$$\begin{array}{r} 2 \quad 12 \end{array}$$

$7 \div 17 = .4$. Diminish the roots by .4.

$$\begin{array}{r} 2 \quad 12. \quad 17. \quad -7 \quad \underline{.4} \\ \quad .8 \quad 5.12 \quad 8.848 \end{array}$$

$$\begin{array}{r} 2 \quad 12.8 \quad 22.12 \quad + \end{array}$$

Since .4 is found to be too large, we try .3.

$$\begin{array}{r} 2 \quad 12. \quad 17. \quad -7 \quad \underline{.3} \\ \quad .6 \quad 3.78 \quad 6.234 \end{array}$$

$$\begin{array}{r} 2 \quad 12.6 \quad 20.78 \quad - \quad .766 \\ \quad .6 \quad 3.96 \end{array}$$

$$\begin{array}{r} 2 \quad 13.2 \quad \quad 24.74 \\ \quad .6 \end{array}$$

$$\begin{array}{r} 2 \quad 13.8 \end{array}$$

$.766 \div 24.74 = .03$.

Diminish the roots by .03.

$$\begin{array}{r}
 2 \quad 13.8 \quad 24.74 \quad - .766 \quad | \quad .03 \\
 \quad .06 \quad .4158 \quad .754674 \\
 \hline
 2 \quad 13.86 \quad 25.1558 \quad | \quad - .011326 \\
 \quad .06 \quad .4176 \\
 \hline
 2 \quad 13.92 \quad 25.5734 \\
 \quad .06 \\
 \hline
 2 \quad | \quad 13.98
 \end{array}$$

$$.011326 \div 25.5734 = .0004.$$

$$\begin{array}{r}
 2 \quad 13.98 \quad 25.5734 \quad - .011326 \quad | \quad .0004 \\
 \quad .0008 \quad .0559232 \quad .01025172928 \\
 \hline
 2 \quad 13.9808 \quad 25.6293232 \quad - .00107427072
 \end{array}$$

Thus the third decimal place is seen to be 0, and the root is -3.3304^+ .

The following suggestions are helpful in special cases.

1. If the root is first located between 0 and 1, it is usually best to obtain the first significant figure by locating the root between two successive tenths, by substituting .1, .2, etc. Then diminish the roots and proceed as before.

2. When you are finding the numerically greater of two roots which have the same sign, the constant term will change sign, usually at the first transformation. This does not mean that you are diminishing by a number that is too large, but simply that the numerically smaller root has had its sign changed by the transformation. This should not cause alarm, as the first figure is definitely known and cannot be too large. It will not occur after the first transformation unless the number by which you are diminishing is too large, except in cases when the equation has two roots which are very nearly equal. This case is best treated by methods of advanced mathematics and such problems will not be considered in this book.

3. Sometimes after the first transformation, division of the constant term by the coefficient of x gives a result greater than 1. In that case diminish by .9. If that is found to be too large, try .8 and so on until the constant term does not change sign.

4. In making the transformations, decimal places beyond the third or fourth may usually be disregarded. This must be done with caution, however.

5. If no decimals have been dropped, and if at any stage of the transformation the constant term becomes 0, the root is rational and has been found exactly.

Why is this true?

6. If at any transformation the coefficient of x becomes 0, we may find the next figure of the root approximately by dividing the numerical term by the coefficient of x^2 and extracting the square root of the result.

Illustrative example III :

Find to three decimal places the roots of the equation $x^3 - 9x + 5 = 0$, without using decimals in transforming.

Solution: This may be accomplished by multiplying the roots of the first transformed equation by 10, the roots of the second transformed equation by 100, etc.

In this example we note that $f(x)$ has but two variations of sign, and therefore cannot have more than two positive roots. $f(-x) = -x^3 + 9x + 5 = 0$. Since it has only one variation of sign it cannot have more than one positive root. Therefore $f(x)$ cannot have more than one negative root.

$f(0) = 5$. We observe that there is a change of sign
 $f(1) = -3$. when x equals 3.
 $f(2) = -5$. We shall, therefore, diminish the roots by 2.
 $f(3) = 5$.

It may be found that the first transformed equation must be reduced by .6, the second by .06, and the third by .001, but by this method we shall reduce them by 6, 6, and 1, respectively. The work may be arranged as follows:

$$\begin{array}{r}
 1 \quad -0 \quad -9 \quad +5 \quad | \quad 2 \\
 \quad \quad 2 \quad +4 \quad -10 \\
 \hline
 1 \quad +2 \quad -5 \quad | \quad -5 \\
 \quad \quad +2 \quad +8 \\
 \hline
 1 \quad +4 \quad | \quad +3 \\
 \quad \quad +2 \\
 \hline
 1 \quad +6 \\
 \hline
 1 \quad +60 \quad +300 \quad -5000 \quad | \quad 6 \\
 \quad \quad +6 \quad +396 \quad +4176 \\
 \hline
 1 \quad +66 \quad +696 \quad | \quad -824 \\
 \quad \quad +6 \quad +432 \\
 \hline
 1 \quad +72 \quad +1128 \\
 \quad \quad +6 \\
 \hline
 1 \quad +78 \\
 \hline
 1 \quad +780 \quad +112800 \quad -824000 \quad | \quad 6 \\
 \quad \quad +6 \quad +4716 \quad +705096 \\
 \hline
 1 \quad +786 \quad +117516 \quad | \quad -118904 \\
 \quad \quad +6 \quad +4752 \\
 \hline
 1 \quad +792 \quad | \quad +122268 \\
 \quad \quad +6 \\
 \hline
 1 \quad +798 \\
 \hline
 1 \quad +7980 \quad +12226800 \quad -118904000 \quad | \quad 9 \\
 \quad \quad +9 \quad +71901 \quad +110688309 \\
 \hline
 1 \quad +7989 \quad +12298701 \quad -8215691
 \end{array}$$

Therefore a root is $2.669 = 2.67^-$.

The above method may be used by those who prefer to eliminate the calculation with decimals.

120. Roots of Numbers. — Horner's method may be used to find any root of any number to any required degree of accuracy. For example if it is required to find the cube root of 73, we have only to find by the above method the positive real root of the equation

$$x^3 = 73, \text{ or } x^3 - 73 = 0.$$

Exercises

Find to the nearest hundredth a positive root of each of the following equations :

$$1. \ x^3 + 7x^2 + x - 6 = 0. \quad 3. \ 4x^3 + 2x^2 - 3 = 0.$$

$$2. \ x^3 + 4x - 8 = 0. \quad 4. \ 2x^4 + x^2 - 10 = 0.$$

Find to 3 decimal places a negative root of each of the following equations :

$$5. \ 2x^3 + x^2 + 3x + 2 = 0. \quad 7. \ x^3 + 2x + 20 = 0.$$

$$6. \ x^3 + 6x^2 + 8 = 0. \quad 8. \ 2x^4 + 6x^2 - 5 = 0.$$

Find to the nearest tenth all the real roots of each of the following equations :

$$9. \ x^4 - 6x^2 + x - 3 = 0. \quad 10. \ x^3 - 2x^2 - x + 6 = 0.$$

$$11. \ x^4 - 4x^3 - 6x^2 + 28x - 7 = 0.$$

$$12. \ x^4 + x^3 - x - 10 = 0.$$

13. Find by Horner's method the positive root of $x^3 + x^2 + x - 100 = 0$ to two places of decimals.

14. Find by Horner's method the positive root of $x^3 + 3x^2 + 5x - 178 = 0$ to two decimal places.

15. Find by Horner's method the positive root, greater than 2, of $x^3 - 3x^2 + 3 = 0$ to two places of decimals.

16. Find the value of the negative root of the equation $x^4 + x^3 + x^2 - 88 = 0$, correct to two decimal places.

17. Find to three decimal places a root of the equation $2x^3 - 6x^2 + 1.92 = 0$, between the limits of 2 and 3.

18. Find to four decimal places, a root of the equation $x^3 + 11x^2 - 102x = -171$ between the limits of 3 and 4.

19. Find to four decimal places, a root of the equation $x^3 - 9x = -9$ between the limits of 1 and 2.

20. Find a root of the equation $2x^3 + 3x^2 = 850$, correct to 4 decimal places between the limits of 7 and 8.

21. Find a root of the equation $5x^3 - 6x^2 + 3x = +85$ lying between 2 and 3, and correct to 4 places of decimals.

Find to the nearest hundredth by Horner's method :

22. $\sqrt[3]{6}$.

23. $\sqrt[3]{-47}$.

24. $\sqrt[4]{30}$.

25. $\sqrt[4]{13.5}$.

26. $\sqrt[5]{52}$.

27. Find the real roots of $x^4 - 3 = 0$, to two decimal places.

28. Solve by Horner's method to two places of decimals :

$$x = \sqrt[3]{3x + 4}.$$

29. A piece of ground 100 feet square is to be raised 1 foot by dirt which is to be taken from a ditch around and bordering the plot of ground. If the ditch is to be as deep as it is wide, what is the depth required?

30. A sphere of ice 1 foot in diameter floating in water sinks to a depth of x feet, given by the equation $2x^3 - 3x^2 + .93 = 0$. What is the depth correct to three decimal places?

31. The number of cubic feet in a cube increased by the number of feet in 6 of its edges equals 100. Find the edge of the cube to two decimal places.

121. Solution of the Cubic Equation by Formula. — Since we have a formula by which any quadratic equation can be solved easily, the question would naturally arise in your minds as to whether there may not be a formula by which

any cubic or higher degree equation can be solved. This question was answered for the cubic equation by Cardan, who published in 1545 the solution which follows and which bears his name. It is now known, however, that it was not original with him, but that he obtained it from the Italian mathematician Tartaglia under promise of secrecy, and afterwards published it as his own.

Cardan's Method for the Solution of Cubics. — The equation $x^3 + p_1x^2 + p_2x + p_3 = 0$ may be reduced to the form $x^3 + px + q = 0$ by a transformation which removes the second term. This may be accomplished by increasing the roots by $\frac{p_1}{3}$. (See section 109.)

Now let $x = m + n$.

And let $3mn = -p$.

This is possible since a number can be divided into two parts having a fixed product.

$$\text{Now} \quad x^3 + px + q = 0. \quad (1)$$

$$\text{Then} \quad (m + n)^3 - 3mnx + q = 0. \quad (2)$$

$$m^3 + 3m^2n + 3mn^2 + n^3 - 3mnx + q = 0. \quad (3)$$

$$m^3 + n^3 + 3mn(m + n) - 3mnx + q = 0. \quad (4)$$

Since $x = m + n$, the third and fourth terms cancel and we have

$$m^3 + n^3 + q = 0. \quad (5)$$

$$\text{But, since } 3mn = -p, \quad m = -\frac{p}{3n}.$$

Hence, substituting in (5),

$$-\frac{p^3}{27n^3} + n^3 + q = 0.$$

$$\text{Or,} \quad 27n^6 + 27qn^3 - p^3 = 0. \quad (6)$$

Hence, by the quadratic formula

$$n^3 = -\frac{27q \pm \sqrt{729q^2 + 108p^3}}{54}.$$

Or,
$$n^3 = -\frac{q}{2} \pm \frac{1}{18}\sqrt{81q^2 + 12p^3}. \quad (7)$$

Likewise
$$m = -\frac{p}{3n},$$

and substituting in (5), we obtain

$$27m^6 + 27qm^3 - p^3 = 0. \quad (8)$$

And
$$m^3 = -\frac{q}{2} \mp \frac{1}{18}\sqrt{81q^2 + 12p^3}. \quad (9)$$

Hence if
$$n = \sqrt[3]{-\frac{q}{2} + \frac{1}{18}\sqrt{81q^2 + 12p^3}},$$

m will
$$= \sqrt[3]{-\frac{q}{2} - \frac{1}{18}\sqrt{81q^2 + 12p^3}}.$$

For the product of m and n is $-\frac{p}{3}$.

But $x = m + n$.

Hence
$$x = \sqrt[3]{-\frac{q}{2} + \frac{1}{18}\sqrt{81q^2 + 12p^3}} + \sqrt[3]{-\frac{q}{2} - \frac{1}{18}\sqrt{81q^2 + 12p^3}}.$$

But every number has 3 cube roots, namely, the real cube root, the real cube root times ω (called omega), and the real cube root times ω^2 , where ω and ω^2 are the imaginary cube roots of unity. See section 46.

The requirement that the product of m and n shall be rational demands that if m is taken as ω times the real cube root of $\left(-\frac{q}{2} + \frac{1}{18}\sqrt{81q^2 + 12p^3}\right)$, then n must be taken as ω^2 times the real cube root of $\left(-\frac{q}{2} - \frac{1}{18}\sqrt{81q^2 + 12p^3}\right)$.

The roots then are

1. $\sqrt[3]{-\frac{q}{2} + \frac{1}{18}\sqrt{81q^2 + 12p^3}}$
 $+ \sqrt[3]{-\frac{q}{2} - \frac{1}{18}\sqrt{81q^2 + 12p^3}}.$
2. $\omega \sqrt[3]{-\frac{q}{2} + \frac{1}{18}\sqrt{81q^2 + 12p^3}}$
 $+ \omega^2 \sqrt[3]{-\frac{q}{2} - \frac{1}{18}\sqrt{81q^2 + 12p^3}}.$
3. $\omega^2 \sqrt[3]{-\frac{q}{2} + \frac{1}{18}\sqrt{81q^2 + 12p^3}}$
 $+ \omega \sqrt[3]{-\frac{q}{2} - \frac{1}{18}\sqrt{81q^2 + 12p^3}}.$

The usefulness of this formula is limited in practice to the solution of the cubic which has one real and two complex roots, except in special cases.

If all the roots are real, the quantities

$$-\frac{q}{2} \pm \frac{1}{18}\sqrt{81q^2 + 12p^3}$$

are imaginary, unless double roots are present, and we have no algebraic method of obtaining the cube root of an imaginary number.

Illustrative example :

Solve by Cardan's formulas :

$$x^3 - 9x^2 + 36x - 28 = 0.$$

Solution: Diminishing the roots by 3, the equation becomes

$$x^3 + 9x + 26 = 0.$$

Then in the above formula, $p = 9$, $q = 26$.

$$\begin{aligned}
 \text{Hence } x &= \sqrt[3]{-13 + \frac{1}{18}\sqrt{54756 + 8748}} \\
 &\quad + \sqrt[3]{-13 - \frac{1}{18}\sqrt{54756 + 8748}} \\
 &= \sqrt[3]{-13 + 14} + \sqrt[3]{-13 - 14} \\
 &= 1 + (-3) = -2.
 \end{aligned}$$

The other roots are $\omega - 3\omega^2$ and $\omega^2 - 3\omega$; that is, $1 + 2\sqrt{-3}$ and $1 - 2\sqrt{-3}$.

But these are the roots of the transformed equation. Hence the roots of the original equation are 1 , $4 + 2\sqrt{-3}$ and $4 - 2\sqrt{-3}$.

Exercises

Solve by Cardan's method each of the following equations :

1. $x^3 + 9x^2 + 12x - 144 = 0$.
2. $x^3 + 6x^2 + 12x + 9 = 0$.
3. $x^3 - 6x^2 + 117x - 436 = 0$.
4. $x^3 - 12x^2 + 3x - 36 = 0$.
5. $x^3 + 9x^2 + 24x + 20 = 0$.
6. $x^3 - 12x^2 + 42x - 49 = 0$.
7. $x^3 - 3x^2 - 6x + 36 = 0$.
8. $x^3 + 3x^2 - 21x + 49 = 0$.
9. $x^3 + 12x^2 + 45x + 50 = 0$.
10. $x^3 - 3x + 2 = 0$.
11. $x^3 - 63x + 370 = 0$.

NOTE. When the expression $81q^2 + 12p^3$ is not a perfect square, the method is cumbersome, and solution by trial, or, if that fails, Horner's method is preferable.

122. Quartic and Higher Equations. — Ferrari, a pupil of Cardan, obtained a general solution of the quartic or biquadratic equation. The method is rather complicated and will not be considered here. For centuries thereafter efforts were

made by mathematicians to find a general solution of the quintic and equations of higher degree, but without success.

Finally the brilliant young Norwegian mathematician Abel proved early in the nineteenth century that it is impossible to express in radical form the roots of the quintic equation or of any equation of degree higher than 5.

HISTORICAL NOTE

Girolamo Cardan was born at Pavia, September 24, 1501, and died at Rome, September 21, 1576. He was a man of furious and ungovernable temper, of the wildest vices, and of great eccentricities. At different times of his life he was devoted to philosophy, to mathematics, to medicine, and to astrology. It was the latter science which led to his death, for having predicted that his death would occur on a certain date, he committed suicide in order that his prediction might be fulfilled.

His mathematical discoveries were published in 1545 in a work called the *Ars Magna*. This was a great advance over the algebras which had been published, containing some discussion of both negative and imaginary roots, which latter he proved to occur in conjugate pairs. It was in this work that he published, as his own, the solution of the cubic equation, which he obtained from Tartaglia under promise of secrecy.

Tartaglia's real name was Niccolo Fontana, but he was called Tartaglia (The Stammerer) because of an impediment in his speech resulting from an injury received in his boyhood. He was born at Brescia in 1500 and died at Venice, December 14, 1557. His father was killed in a war with the French in 1512, and the family were left in direst poverty. Niccolo had but fifteen days of schooling, but that was enough to stir his ambition, so that he became self-educated. It is said that he used the tombstones in the local cemetery as slates on which to perform his problems in mathematics, since he was too poor to buy paper. He became a very famous teacher, and the author of at least three important mathematical works.

Ferrari, whose real name was Ludovico Ferraro, was one of the most famous of the pupils of Cardan. He was born February 2, 1522 and died October 5, 1565. He first entered Cardan's service as an errand boy, but became such an apt pupil that he exceeded his master, at least in having accomplished the solution of the quartic equation in which Cardan had failed. His last days were clouded by poor health, and he was finally poisoned by his sister.

Niels Henrik Abel was born at Findoe, Norway, August 5, 1802. He gave promise of becoming one of the greatest mathematicians of all time, but unfortunately he died April 6, 1829, when only 26 years of age. In that short life he produced some monumental works. His investigation was chiefly connected with the roots of equations, and led to results which are now usually included under the "Theory of Substitutions." He was the first to prove that an algebraic solution of the general quintic or higher degree equation is impossible. He also proved the truth of the binomial theorem for the expansion of $(a + b)^n$, where n is a complex number.

Cumulative Review

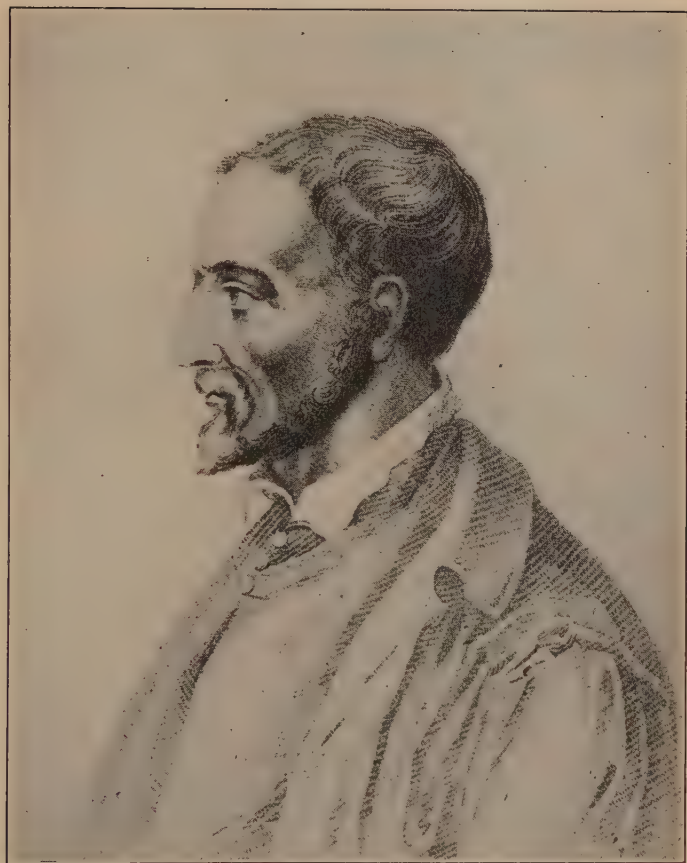
1. Show that if $a + ib$ is a root of an equation with real coefficients that $a - ib$ is also a root.

2. One root of $x^4 - 3x^2 - 42x - 40 = 0$ is $-\frac{1}{2}(3 - \sqrt{-31})$. Find the other roots.

3. Graph $y = x^3 + 3x^2 - 2x - 5$ between the values $x = -4$ and $x = 2$. From the figure determine between what consecutive integers the roots of the equation $x^3 + 3x^2 - 2x - 5 = 0$ lie.

4. Solve by Cardan's method: $x^3 - 3x^2 - 6x - 20 = 0$.

5. Find by trial, the six roots of $x^6 + 5x^5 - 81x^4 - 85x^3 + 964x^2 + 780x - 1584 = 0$.



GIROLAMO CARDAN

6. In the equation $x^4 + 7x^3 + x^2 - 63x - 90 = 0$ the sum of two of the roots is -8 , and the product of the other two is -6 . Find the roots.

7. Find the upper limit to the number of positive and negative real roots of $x^4 - 3x + 2 = 0$. Find also the fewest possible imaginary roots.

8. Transform the following into an equation whose second term shall be lacking:

$$x^4 - 8x^3 - 5x + 1 = 0.$$

9. If one of the roots of the equation $x^4 + 12x^3 + 78x^2 + 252x + 272 = 0$ is $-3 + 5i$, find the other roots.

10. Draw the graph of $y = x^3 - 4x^2 - 4x + 12$. Locate the roots of $x^3 - 4x^2 - 4x + 12 = 0$ between consecutive integers and find the larger positive one to the nearest hundredth.

11. Show by the principles of permutations and combinations how many distinct lines can be drawn through twelve points, no three of which lie in the same straight line.

12. In how many ways can a committee of 3 teachers and 2 students be formed from 5 teachers and 15 students?

13. Transform the equation $x^4 + \frac{1}{3}x^3 + \frac{2}{5}x^2 + \frac{2}{25} = 0$ into an equation having only integral coefficients, that of the first term being unity. How do the roots of the derived equation differ from the roots of the given equation?

14. (a) Reduce $\frac{2 + 3i}{3 - 4i}$ and $\frac{3 + 2i}{3 + 4i}$ to the form $a + bi$.

(b) Construct graphically the sum and difference of $3 - 4i$ and $-5 + 6i$.

(c) Express $\left(\frac{1+i}{2}\right)^3$ and $\sqrt{7+24i}$ in the form $a + bi$.

15. How many combinations of 3 strings can be struck on an instrument of 8 strings?

16. What information, if any, do Descartes' Rules of Signs give in regard to the roots of the equation $2x^3 - 7x^2 + 3x - 1 = 0$?

Find one root of this equation correct to three significant figures.

17. Find and simplify the ninth term in the expansion of

$$\left(a^2 - \frac{2}{a^{\frac{1}{2}}}\right)^{12}.$$

18. (a) Find the sum of ten terms of an arithmetical progression whose first term is a and whose second term is b .

(b) The second term of a geometrical progression is 3 and the fourth term is $\frac{1}{3}$. Find the ninth term.

19. If $x = at^3$ and $y = a\sqrt{(1+t^2)^3}$, find the value of $x^{\frac{2}{3}} + y^{\frac{2}{3}}$, reducing the answer to its simplest form.

20. (a) Insert three geometric means between 6 and $\frac{32}{7}$.

(b) Insert six arithmetic means between 7 and 35.

21. What information, if any, do Descartes' Rules of Signs give in regard to the roots of the equation

$$x^3 - 4x^2 - 3x + 19 = 0?$$

22. Find the values of x and y in

$$x^2 + y^2 + x + y = 8$$

and

$$xy + x + y = 5.$$

23. There are four numbers of which the first three are in arithmetic progression and the last three in geometric progression. Their sum is 21, and the third minus the first is 4. Find the numbers.

24. (a) Simplify $\sqrt[3]{\frac{a^{-1}}{b^{-3}}} \div \left(\frac{a^{-\frac{2}{3}}b^{\frac{1}{2}}}{b^{-1}a}\right)^2$.

(b) Simplify $(\sqrt{x} + \sqrt[4]{x} + 1)(x^{-\frac{1}{2}} + x^{-\frac{1}{4}} + 1)$.

25. Extract the square root of $46 + 7\sqrt{-12}$.

26. Solve $\sqrt{10x - 34} + 2\sqrt{x + 4} = \sqrt{2(3x + 35)}$.

27. If a and b are the roots of the equation $hx^2 + px + q = 0$, find the equation whose roots are $(a + b)^2$ and $(a - b)^2$.

28. Two rectangles have the same altitude $\sqrt{189}$, and their bases differ by 4. If the difference of the diagonals is 2, find the bases of the rectangles.

29. Draw the graph of $y = 9x^4 + 2x^3 - 3x^2 + 0.1$ and from this graph locate the real roots of $9x^4 + 2x^3 - 3x^2 + 0.1 = 0$.

30. Find the area of a square whose diagonal is 1 foot longer than a side.

31. A and B are two stations 300 miles apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B in nine hours, the train from B reaches A four hours after they meet. Find the rate at which each train travels.

32. Given $a : b = c : d$;

prove that $(a + b) : (c + d) = \sqrt{a^2 + b^2} : \sqrt{c^2 + d^2}$.

CHAPTER X

DETERMINANTS

IN this chapter we shall explain a very concise and convenient method of notation, which you will find useful not only in the solution of certain types of systems of equations, but also in your later work in mathematics.

123. The Determinant. — If we solve the system of linear equations $\left. \begin{array}{l} ax + by = c \\ mx + ny = p \end{array} \right\}$, we obtain as solutions $x = \frac{cn - bp}{an - bm}$ and $y = \frac{ap - cm}{an - bm}$.

Algebraic expressions of a type similar to these numerators and denominators are of frequent occurrence, and it is convenient to symbolize them thus,

$$x = \frac{\begin{vmatrix} c & b \\ p & n \end{vmatrix}}{\begin{vmatrix} a & b \\ m & n \end{vmatrix}}, \quad \text{and} \quad y = \frac{\begin{vmatrix} a & c \\ m & p \end{vmatrix}}{\begin{vmatrix} a & b \\ m & n \end{vmatrix}}.$$

In this form the square array of numbers with bars down the sides which appear as numerator and denominator of our result are called *determinants*. The original or expanded form can be obtained by taking the product of the diagonal quantities from the upper left corner to the lower right corner as positive and the product of the quantities forming the other diagonal as negative in each case. We are now prepared to define a determinant thus:

Such a symbol as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, consisting of n^2 quantities

arranged in a square array of n rows (horizontal lines) and n columns (vertical lines), and with vertical bars down the sides, is called the *determinant* of those quantities. The quantities $a_1, a_2, b_1, b_2, c_1, c_2$, etc., are called the *elements* of the determinant. In expanded form, it represents the algebraic sum of all the products which can be formed by taking one and only one quantity from each row and one and only one from each column. The sign of each product is determined by the following

Rule. — (1) *Arrange the quantities which make up the product in order, according to the columns from which they are obtained.*

(2) *Then the subscript of each element indicates the row from which it is obtained. Count the number of inversions of order among these subscripts.*

(3) *If the number of inversions is even the sign of that product is +, and if the number of inversions is odd the sign is -.*

For example, one such product in the expansion of the above determinant is $a_2c_1b_3$. Arranging the elements in order according to columns, we have $a_2b_3c_1$. Here 2 comes before 1, and 3 comes before 1, hence there are two inversions of order and the sign of this product is +.

In a similar manner, we might determine all the products and their signs. The expansion of the above determinant will be found to be $a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_1a_2c_3 - a_1b_3c_2$.

124. Order of Determinant. — A determinant which consists of n rows and n columns is said to be of the n th order.

Example: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a determinant of the second order.

125. General Theorem Concerning Determinants. —

Theorem. — *Each term in the expansion of a determinant is the product of n factors.*

This follows at once since there is one and only one quantity taken from each column, and the number of columns is n .

Theorem. — *The number of terms in the expansion is $n!$.*

Proof: The terms of the expansion when arranged according to columns contain the n subscripts arranged in every possible order. Hence there must be $n!$ terms.

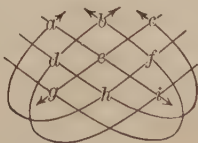
126. Expansion of a Determinant of Third Order. — In order to obtain a determinant in expanded form it is not necessary to determine the products and signs by the rule of section 123. We shall now explain a simpler process.

Consider the second order determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

The diagonal line a_1b_2 is called the *principal diagonal*, and the diagonal line b_1a_2 is called the *secondary diagonal*. The product of the elements in the principal diagonal is $+$, and the product of those of the secondary diagonal $-$.

$$\text{Hence } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2.$$

An extension of this process may be used to expand a determinant of the third order, such as



Here the product of the elements in the principal diagonal aei is positive, as are the products of all lines parallel to it, as indicated by the lines in the diagram, namely, bfg and dhc .

In the same manner, the product of the quantities forming the secondary diagonal ceg is negative, as well as the parallel lines fha and idb .

Hence
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \equiv aei + bfg + dhc - ceg - fha - bdi.$$

NOTE. Attention must be paid to the sign of each element of the determinant.

Illustrative example :

Evaluate the determinant
$$\begin{vmatrix} 2 & 1 & -4 \\ 3 & -1 & 0 \\ 5 & 2 & 3 \end{vmatrix}.$$

Solution : By the method of the preceding section

$$\begin{vmatrix} 2 & 1 & -4 \\ 3 & -1 & 0 \\ 5 & 2 & 3 \end{vmatrix} = 2(-1)3 + 1 \cdot 0 \cdot 5 + 3 \cdot 2 \cdot (-4) - (-4) \\ (-1) \cdot 5 - 1 \cdot 3 \cdot 3 - 2 \cdot 0 \cdot 2 \\ = -6 + 0 - 24 - 20 - 9 - 0 = -59$$

Exercises

Expand each of the following determinants :

1. $\begin{vmatrix} x & a \\ y & b \end{vmatrix}.$
2. $\begin{vmatrix} x^2 & -a^2 \\ -y^2 & b^2 \end{vmatrix}.$
3. $\begin{vmatrix} a & l & k \\ t & m & n \\ r & s & v \end{vmatrix}.$
4. $\begin{vmatrix} k & 1 & 1 \\ 0 & k & 1 \\ 0 & 0 & k \end{vmatrix}.$
5. $\begin{vmatrix} a & b & a \\ b & a & b \\ a & b & a \end{vmatrix}.$
6. $\begin{vmatrix} a & b & c \\ -b & a & b \\ -c & -b & a \end{vmatrix}.$
7. $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 2 & 5 \\ 4 & 1 & 3 \end{vmatrix}.$
8. $\begin{vmatrix} -3 & 2 & -3 \\ 0 & -1 & 4 \\ 5 & 6 & -2 \end{vmatrix}.$
9. $\begin{vmatrix} -1 & 4 & 6 \\ -2 & 5 & 0 \\ -3 & -1 & -2 \end{vmatrix}.$
10. $\begin{vmatrix} 2 & 4 & 1 \\ -1 & -2 & 5 \\ 3 & 6 & 9 \end{vmatrix}.$
11. $\begin{vmatrix} a & d & (a-d) \\ b & e & (b-e) \\ c & f & (c-f) \end{vmatrix}.$

Write as a determinant each of the following :

12. $x^2 - ab$.

13. $x^3 + bdf + ace - bxe - adx - cfx$.

14. Show by actual expansion that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ m & n & p \end{vmatrix} = \begin{vmatrix} a & x & m \\ b & y & n \\ c & z & p \end{vmatrix}.$$

15. Evaluate $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 7 & 4 \\ 2 & 2 & 3 \end{vmatrix}$ and state fully how the signs of

the terms of the development are determined.

16. Expand and reduce $\begin{vmatrix} x + y & x - y \\ x + y & x - y \end{vmatrix}$.

Expand and reduce :

17. $\begin{vmatrix} x - 2z - y^2 \\ -y - 2x & z^2 \\ -z & 2y - x^2 \end{vmatrix}$.

19. $\begin{vmatrix} a - 2a & b \\ 3b - c & 4d \\ 2c & 3d - 4b \end{vmatrix}$.

18. $\begin{vmatrix} a + b & c & c \\ a & b + c & a \\ b & b & c + a \end{vmatrix}$.

20. $\begin{vmatrix} 1 + x & 1 & 1 \\ 1 & 1 + y & 1 \\ 1 & 1 & 1 + z \end{vmatrix}$.

127. Theorem. — *A determinant is unaltered if rows are changed into columns and columns into rows.*

Proof: This is obviously true since elements of rows and columns enter in the same way into each product. Hence, any theorem which states a fact concerning columns is equally true concerning rows and conversely.

Example : $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ when expanded.

128. Simplification of Determinants. — The evaluation of determinants may be simplified by use of section 127 as well as by the following theorems.

Theorem 1. *If all the elements of any row or column are 0, the value of the determinant is 0.*

For an element from that row or column occurs as a factor of each term.

Theorem 2. — *If every element of any row or column be multiplied by a constant k , the entire determinant is multiplied by that constant.*

Proof: Every term in the expansion contains an element from this row or column as a factor. In a certain term let the element be a . Then after the elements of the row or column all have been multiplied by k , this term will contain the factor ka . Hence the term will have been multiplied by k . In like manner every other term is multiplied by k and hence the whole determinant is multiplied by k .

$$\text{Compare } k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{with} \quad \begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix}.$$

The principal diagonal of the latter is $ka_1b_2c_3$. By expanding the latter determinant and factoring out k we obtain

$$k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Corollary I. — *If every element of any row or column be divided by a constant k , the entire determinant is divided by that constant.*

For division by k is the same as multiplication by $\frac{1}{k}$.

Corollary II. — *If a constant k can be removed as a factor from all the elements of any row or column, then k is a factor of the determinant.*

$$\text{For example, } \begin{vmatrix} ka & c \\ kb & d \end{vmatrix} = k \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

Theorem 3. — *If two rows or columns in a determinant are interchanged, the sign of the determinant is changed.*

Proof: When any term in the expansion has its elements arranged according to columns, such an interchange will interchange two of the subscripts which indicate the order of the elements, as determined by rows, thus increasing or diminishing by one the number of inversions of order among them. Hence the sign of this term is changed. Similarly the sign of every term is changed, thus changing the sign of the determinant.

For example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}.$$

Theorem 4. — *If any row or column is moved over n other rows or columns, the sign of the determinant will be changed if n is odd, and unchanged if n is even.*

This theorem follows at once when we consider that such a transformation may be effected by n successive interchanges with adjacent rows or columns.

Theorem 5. — *In the expansion of a determinant half the terms are positive and half negative.*

Proof: If two subscripts are interchanged in any term, another term of the expansion is obtained, since the expansion contains terms with these subscripts arranged in every possible order. Such an interchange will change the sign of the term. Hence, there must be at least as many negative terms as positive terms, and at least as many positive terms as negative terms. There must, therefore, be the same number of each.

Theorem 6. — *If each element of a row or column can be expressed as the sum of two quantities, then the determinant can be expressed as the sum of two determinants.*

This theorem may be made clear by the following illustration.

$$\begin{vmatrix} a+b & x \\ c+d & y \end{vmatrix} = \begin{vmatrix} a & x \\ c & y \end{vmatrix} + \begin{vmatrix} b & x \\ d & y \end{vmatrix}.$$

$$\begin{aligned} \text{For } \begin{vmatrix} a+b & x \\ c+d & y \end{vmatrix} &= (a+b)y - (c+d)x \\ &= ay + by - cx - dx \\ &= (ay - cx) + (by - dx) \\ &= \begin{vmatrix} a & x \\ c & y \end{vmatrix} + \begin{vmatrix} b & x \\ d & y \end{vmatrix}. \end{aligned}$$

Theorem 7. — *If a determinant has two rows or columns identical, it is equal to zero.*

Proof: Let the value of the determinant be denoted by D . Then let the two like rows or columns be interchanged. By a preceding theorem, the sign of the determinant is changed. However, the interchange of two *identical* rows or columns cannot change the value of the determinant.

$$\text{Hence,} \quad D = -D.$$

$$\therefore 2D = 0$$

and

$$D = 0.$$

Theorem 8. — *If the elements of one row or column of a determinant are equal to the corresponding elements of another row or column multiplied by a constant, the determinant is equal to zero.*

Proof: The determinant

$$\begin{vmatrix} a_1 & ma_1 & b_1 \\ a_2 & ma_2 & b_2 \\ a_3 & ma_3 & b_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = m \cdot 0 = 0.$$

Theorem 9. — *If all the elements of any row or column are replaced by a row or column, which is made by adding to those elements equimultiples of the corresponding elements of any other row or column, the value of the determinant is unchanged.*

To prove
$$\begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Proof:

$$\begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ since the determinant } \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} = 0.$$

129. Determinants by Minors. — If a row and column of a determinant of order n are deleted, the resulting determinant of order $(n - 1)$ is called the **minor** of that element which is twice deleted. For example, if we delete the first row and

column of the determinant $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$ thus $\begin{vmatrix} e & f \\ h & k \end{vmatrix}$, then the determinant $\begin{vmatrix} e & f \\ h & k \end{vmatrix}$ is the *minor* of the element a . In like manner $\begin{vmatrix} b & c \\ h & k \end{vmatrix}$ is the *minor* of d , and $\begin{vmatrix} a & c \\ g & k \end{vmatrix}$ is the minor of e , and so on.

130. Expansion of a Determinant by Minors. — The processes already given for expanding a determinant apply only to examples of the second and third order. The following method is general.

Rule. — (1) *Select any row or column as the one in terms of whose minors the determinant is to be expanded.*

(2) *Write each element of this row or column multiplied by its respective minor.*

(3) *Give to each a sign determined by the number of shifts of position necessary to move the coefficient to prime position, that is, to the upper left-hand corner of the determinant.*

(4) *An even number of shifts produces the + sign, and odd number the - sign.*

You should study with care the following illustrations of these various principles.

Illustrative example I:

Expand in terms of minors of the second row.

$$\begin{vmatrix} a & h & k \\ r & s & v \\ k & m & n \end{vmatrix}.$$

Solution: r requires one shift, s , two shifts, and v , three shifts to move them to prime position. Hence the terms having these coefficients are respectively negative, positive, and negative.

$$\begin{aligned} \text{Hence } \begin{vmatrix} a & h & k \\ r & s & v \\ k & m & n \end{vmatrix} &= -r \begin{vmatrix} h & k \\ m & n \end{vmatrix} + s \begin{vmatrix} a & k \\ k & n \end{vmatrix} - v \begin{vmatrix} a & h \\ k & m \end{vmatrix} \\ &= -r(hn - km) + s(an - k^2) - v(am - hk) \\ &= -rhn + rkm + san - sk^2 - vam + vhk. \end{aligned}$$

In a similar manner, a determinant of the fourth order may be expanded into 4 determinants of the third order, and each one of these into 3 determinants of the second order, and similarly for a determinant of any order.

Illustrative example II :

$$\text{Evaluate } \begin{vmatrix} 9 & 1 & 4 & 0 \\ 14 & 2 & 6 & 2 \\ 11 & 5 & 3 & 5 \\ 5 & 1 & 2 & 7 \end{vmatrix}.$$

Solution: Applying the last theorem of section 128, we subtract the elements of the second column from those of

$$\text{the first. We obtain } \begin{vmatrix} 8 & 1 & 4 & 0 \\ 12 & 2 & 6 & 2 \\ 6 & 5 & 3 & 5 \\ 4 & 1 & 2 & 7 \end{vmatrix}. \quad \text{This determinant}$$

equals zero, since the first column is twice the third column. See theorem 8, section 128.

$$\text{Hence } \begin{vmatrix} 9 & 1 & 4 & 0 \\ 14 & 2 & 6 & 2 \\ 11 & 5 & 3 & 5 \\ 5 & 1 & 2 & 7 \end{vmatrix} = 0.$$

Illustrative example III :

$$\text{Evaluate } \begin{vmatrix} 1 & 1 & 0 & 3 \\ 4 & 3 & 6 & 2 \\ 3 & 5 & 9 & 0 \\ 2 & 1 & 3 & 7 \end{vmatrix}.$$

Solution: Again applying the last theorem of section 128, we proceed as follows. Subtract the elements of the first column from those of the second column, and three times the elements of the first column from those of the fourth column.

$$\text{This gives } \begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & -1 & 6 & -10 \\ 3 & 2 & 9 & -9 \\ 2 & -1 & 3 & 1 \end{vmatrix}.$$

If now we expand this determinant in terms of elements of

the first row, we obtain only $1 \begin{vmatrix} -1 & 6 & -10 \\ 2 & 9 & -9 \\ -1 & 3 & 1 \end{vmatrix}$ since the other

minors have zero coefficients. This was the purpose of the simplification by which we obtained a row with as many zeros as possible. Now add twice the elements of the first row to those of the second row, and subtract the elements of the first row from those of the third row.

$$\begin{aligned} \text{We obtain } \begin{vmatrix} -1 & 6 & -10 \\ 0 & 21 & -29 \\ 0 & -3 & 11 \end{vmatrix} &= -1 \begin{vmatrix} 21 & -29 \\ -3 & 11 \end{vmatrix} \\ &= -(231 - 87) = -144. \end{aligned}$$

Illustrative example IV:

$$\text{Show that } \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a).$$

Solution: Removing the factors a , b , and c from their respective columns, we obtain $abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$. Now subtract

the elements of the first column from those of the second and third columns.

$$\begin{aligned} \text{We now have } abc \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= abc \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= abc(b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} \\ &= abc(b-a)(c-a)(c+a-b-a) \\ &= abc(a-b)(b-c)(c-a). \end{aligned}$$

Exercises

Expand by minors of the first column each of the following determinants:

$$1. \begin{vmatrix} s & k & l \\ r & u & x \\ t & s & r \end{vmatrix}.$$

$$3. \begin{vmatrix} 1 & 4 & 3 \\ 3 & 7 & 9 \\ 5 & 10 & 11 \end{vmatrix}.$$

$$2. \begin{vmatrix} a^2 & x^2 & z^2 \\ m & n & t \\ -a & -e & -r \end{vmatrix}.$$

$$4. \begin{vmatrix} 0 & a & b & c \\ -a & 0 & b & c \\ -a & -b & 0 & c \\ -a & -b & -c & 0 \end{vmatrix}.$$

Solve the following equations:

$$5. \begin{vmatrix} 3-2x & 5 \\ 3 & 1-x \end{vmatrix} = 0.$$

$$6. \begin{vmatrix} 2-x & 3 & 5 \\ 5+3x & 2 & 7 \\ 2+x & 1 & 9 \end{vmatrix} = 0.$$

Simplify and evaluate:

$$7. \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 4 \\ 2 & -1 & 3 \end{vmatrix}.$$

$$11. \begin{vmatrix} 2 & 7 & 0 & -1 \\ 3 & 4 & 2 & 3 \\ 0 & 5 & 8 & -2 \\ 1 & 3 & 6 & 1 \end{vmatrix}.$$

$$8. \begin{vmatrix} 5 & 6 & 4 \\ 4 & 2 & 0 \\ 9 & 3 & 5 \end{vmatrix}.$$

$$12. \begin{vmatrix} -1 & 3 & 6 & 7 \\ 3 & -5 & 2 & 0 \\ 0 & 2 & -1 & 4 \\ 10 & 8 & 3 & -2 \end{vmatrix}.$$

$$9. \begin{vmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \\ 4 & 6 & 8 & 10 \end{vmatrix}.$$

$$13. \begin{vmatrix} b & 1 & 1 & 1 \\ 1 & b & 1 & 1 \\ 1 & 1 & b & 1 \\ 1 & 1 & 1 & b \end{vmatrix}.$$

$$10. \begin{vmatrix} 1 & 6 & 3 & 0 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 3 \\ 5 & 4 & 8 & 1 \end{vmatrix}.$$

14. How many terms are there in the expansion of a determinant of the tenth order? Each term is the product of how many factors?

Prove each of the following identities :

$$15. \begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix} = (y-z)(z-x)(x-y)(x+y+z).$$

$$16. \begin{vmatrix} x & 3 & y \\ x^2 & 9 & y^2 \\ x^3 & 27 & y^3 \end{vmatrix} = 3xy(x-3)(y-3)(x-y).$$

$$17. \begin{vmatrix} m-n & n-p & p-m \\ n-p & p-m & m-n \\ p-m & m-n & n-p \end{vmatrix} = 0.$$

131. Co-Factors. — In the expansion of the determinant in terms of minors of any row or column, the co-factor of any element of that row or column is its minor with proper sign.

For example, in expanding $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ in terms of minors of

the first column, we obtain $a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$

Hence the co-factors of a_1 , a_2 , and a_3 are respectively

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \text{ and } \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

If now we denote these co-factors of a_1 , a_2 , and a_3 by A_1 , A_2 , and A_3 respectively, the expansion of the above determinant may be written

$$a_1 A_1 + a_2 A_2 + a_3 A_3.$$

Notice that the elements which compose the quantities A_1 , A_2 , and A_3 are those which make up the second and third columns. It follows that both $b_1 A_1 + b_2 A_2 + b_3 A_3$ and $c_1 A_1 + c_2 A_2 + c_3 A_3$ are equal to zero, since either represents the expansion of a determinant having two columns identical.

132. Systems of Equations Solved by Determinants. —

Consider the system of linear equations $\begin{cases} 2x + 3y = 13. \\ 4x - y = 5. \end{cases}$

This system may be solved by determinants as follows :

$$x = \frac{\begin{vmatrix} 13 & 3 \\ 5 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} 2 & 13 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}}.$$

Notice that the determinants which form the denominator are the same, and in both cases the determinant is formed by the coefficients of the x and y terms in the order in which they stand. The determinant in the numerator in each case is the same as that in the denominator except that the numerical terms replace the coefficients of the letter for which we are solving.

Simplifying the above, we obtain

$$x = \frac{-13 - 15}{-2 - 12} = \frac{-28}{-14} = +2.$$

$$y = \frac{10 - 52}{-2 - 12} = \frac{-42}{-14} = +3.$$

Now let us consider the general system

$$a_1x + b_1y + c_1z = d_1.$$

$$a_2x + b_2y + c_2z = d_2.$$

$$a_3x + b_3y + c_3z = d_3.$$

Multiply the first equation by A_1 where A_1 is the co-factor of a_1 in the determinant of the coefficients,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ as explained in the preceding section. Like-}$$

wise multiply the second equation by A_2 and the third equation by A_3 . We obtain

$$a_1A_1x + b_1A_1y + c_1A_1z = d_1A_1.$$

$$a_2A_2x + b_2A_2y + c_2A_2z = d_2A_2.$$

$$a_3A_3x + b_3A_3y + c_3A_3z = d_3A_3.$$

Adding, we obtain

$$(a_1A_1 + a_2A_2 + a_3A_3)x + (b_1A_1 + b_2A_2 + b_3A_3)y + (c_1A_1 + c_2A_2 + c_3A_3)z = (d_1A_1 + d_2A_2 + d_3A_3).$$

The coefficient of x is the expansion of the determinant of the coefficient; the coefficients of y and z are zero; the right member of the equation represents the expansion of the same determinant with the column of d 's replacing the column of a 's.

$$\text{Hence } x = \frac{d_1A_1 + d_2A_2 + d_3A_3}{a_1A_1 + a_2A_2 + a_3A_3} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

In a similar manner if we multiply these equations by B_1 , B_2 , B_3 respectively and add, we shall eliminate x and z and obtain

$$y = \frac{d_1B_1 + d_2B_2 + d_3B_3}{b_1B_1 + b_2B_2 + b_3B_3} = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Similarly

$$z = \frac{d_1C_1 + d_2C_2 + d_3C_3}{c_1C_1 + c_2C_2 + c_3C_3} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Hence to solve systems of equations by determinants, we have the following

Rule. — (1) *Write the equations in standard form with numerical terms in the right members.*

(2) *Write the value of each unknown as the quotient of two determinants.*

(3) *The denominator determinant in each case is the determinant formed by the coefficients of the unknowns in the order in which they stand.*

(4) *The numerator determinant is the same determinant as in the denominator except that the numerical terms replace the coefficient of the letter for which you are solving.*

(5) *Simplify and evaluate each determinant.*

Illustrative example :

Solve by determinants

$$x + 3y + 2z = -3.$$

$$3x + y - 5z = 19.$$

$$4x - 2y + z = -12.$$

Solution :

$$x = \frac{\begin{vmatrix} -3 & 3 & 2 \\ 19 & 1 & -5 \\ -12 & -2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 2 \\ 3 & 1 & -5 \\ 4 & -2 & 1 \end{vmatrix}} = \frac{\begin{vmatrix} -60 & 0 & 17 \\ 19 & 1 & -5 \\ 26 & 0 & -9 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 2 \\ 0 & -8 & -11 \\ 0 & -14 & -7 \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} -60 & 17 \\ 26 & -9 \end{vmatrix}}{\begin{vmatrix} -8 & -11 \\ -14 & -7 \end{vmatrix}} = \frac{+540 - 442}{56 - 154} = \frac{98}{-98} = -1.$$

$$y = \frac{\begin{vmatrix} 1 & -3 & 2 \\ 3 & 19 & -5 \\ 4 & -12 & 1 \end{vmatrix}}{-98} = \frac{\begin{vmatrix} 1 & -3 & 2 \\ 0 & 28 & -11 \\ 0 & 0 & -7 \end{vmatrix}}{-98}$$

$$= \frac{\begin{vmatrix} 28 & -11 \\ 0 & -7 \end{vmatrix}}{-98} = \frac{-196}{-98} = 2.$$

$$z = \frac{\begin{vmatrix} 1 & 3 & -3 \\ 3 & 1 & 19 \\ 4 & -2 & -12 \end{vmatrix}}{-98} = \frac{\begin{vmatrix} 1 & 3 & -3 \\ 0 & -8 & 28 \\ 0 & -14 & 0 \end{vmatrix}}{-98}$$

$$= \frac{\begin{vmatrix} -8 & 28 \\ -14 & 0 \end{vmatrix}}{-98} = \frac{392}{-98} = -4.$$

Hence, $x = -1$.

$$y = 2.$$

$$z = -4.$$

These values will check in the three original equations.

Exercises

Solve by determinants each of the following systems of equations:

1. $x + 2y = 8.$

$3x + y = 9.$

3. $4x + 3y = 7.$

$5x - 2y = 26.$

2. $x + y = 7.$

$2x + 3y = 18.$

4. $x + 4y = 10.$

$2x + 3y = 5.$

5. $8x - 5y = 1.$

$\frac{11x - 5y}{22} = \frac{3x + y}{32}.$ *Hint: First clear of fractions.*

$$\begin{aligned} 6. \quad .2x + .7y &= 5. \\ .6x + .6y &= 6. \end{aligned}$$

$$\begin{aligned} 7. \quad 4x + y + z &= 0. \\ x - y - 3z &= 9. \\ 3x + 2y - 4z &= 7. \end{aligned}$$

$$\begin{aligned} 8. \quad x - y + 5z &= 3. \\ 2x + 3y - 2z &= 21. \\ -x + 4y - z &= 6. \end{aligned}$$

$$\begin{aligned} 9. \quad x + y - z &= 0. \\ 2x - y + 2z &= 11. \\ 3x - 2y - 3z &= -15. \end{aligned}$$

$$\begin{aligned} 10. \quad x + 2y - 4z &= -2. \\ 3x + 5y + 2z &= 2. \\ 2x - 7y - 6z &= 8. \end{aligned}$$

$$\begin{aligned} 11. \quad 7x - 3y + z &= 0. \\ 2x + 3y - 5z &= -16. \\ x + y - 6z &= -9. \end{aligned}$$

$$\begin{aligned} 12. \quad 4x + 8y + 3z &= 5. \\ 2x - 4y + 6z &= 2. \\ 10x - 12y - 9z &= -1. \end{aligned}$$

$$13. \quad \frac{x}{3} + \frac{3y}{2} - z = 11\frac{1}{6}.$$

$$2x - 3y + \frac{5z}{2} = -22.$$

$$3x + \frac{y}{4} - 2z = 7.$$

$$\begin{aligned} 14. \quad \frac{2x}{3} + y - 2z &= 15\frac{2}{3}. \\ 3x - 5y + z &= -28. \\ -2x - y + 3z &= -20. \end{aligned}$$

$$\begin{aligned} 15. \quad x - y + 3z + 2w &= 9. \\ 3x + y - z - 3w &= -7. \\ 4x + 2y + 5z - \frac{1}{2}w &= -5. \\ -x + y - z + 4w &= 3. \end{aligned}$$

$$\begin{aligned} 16. \quad x + 5y &= 13. \\ 3x - z &= 13. \\ 5y + 2z &= 2. \end{aligned}$$

$$\begin{aligned} 17. \quad 3x - 2y &= 18. \\ x + 4z &= 0. \\ 5y - 8z &= -7. \end{aligned}$$

18. Determine by what numbers the following equations must be multiplied respectively in order that we may eliminate y and z simultaneously by addition. Using that method, solve the system for x .

$$\begin{aligned} 3x + 2y + z &= 1. \\ -x + y - 5z &= -3. \\ 6x - 5y + z &= -36. \end{aligned}$$

133. Homogeneous Systems. — Let us consider the homogeneous system

$$\begin{aligned} 3x + y + 9z &= 0. \\ x - y - z &= 0. \\ 2x - 3y - 5z &= 0. \end{aligned}$$

It is obvious that $x = 0, y = 0, z = 0$ will satisfy this system. It is equally plain that in solving it by determinants, the numerator determinant will be zero since it will contain a column of zeros, and the same thing is true of any system of linear equations in which numerical terms are zero. Hence $x = 0, y = 0, z = 0$, will always be roots of such a system. This solution is known as *the zero solution*.

Sometimes such systems have other solutions besides the zero solution, but this can be true only when the denominator determinant is also zero, since zero divided by zero may have any finite value whatever. Hence the

Principle: *A system of homogeneous linear equations has a solution other than the zero solution only when the determinant of the coefficient vanishes.*

The determinant of the coefficients in the above gives

$$\begin{vmatrix} 3 & 1 & 9 \\ 1 & -1 & -1 \\ 2 & -3 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 12 \\ 1 & 0 & 0 \\ 2 & -1 & -3 \end{vmatrix} = - \begin{vmatrix} 4 & 12 \\ -1 & -3 \end{vmatrix} = 0.$$

Hence this determinant has another solution than the zero solution. To obtain it we proceed as follows.

Divide each of the equations by z , then

$$\begin{aligned} 3 \frac{x}{z} + \frac{y}{z} &= -9. \\ \frac{x}{z} - \frac{y}{z} &= 1. \\ 2 \frac{x}{z} - 3 \frac{y}{z} &= 5. \end{aligned}$$

Now regarding $\frac{x}{z}$ and $\frac{y}{z}$ as unknowns, we solve the system formed by the first two equations, using determinants or otherwise.

$$\text{Then } \frac{x}{z} = \frac{\begin{vmatrix} -9 & 1 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{9 - 1}{-3 - 1} = -2$$

$$\text{and } \frac{y}{z} = \frac{\begin{vmatrix} 3 & -9 \\ 1 & 1 \end{vmatrix}}{-4} = \frac{3 + 9}{-4} = -3.$$

These values are found by trial to satisfy the third equation.

$$\text{Hence } \frac{x}{z} = \frac{-2}{1}, \text{ and } \frac{y}{z} = \frac{-3}{1}.$$

$\therefore x : y : z = 2 : 3 : -1$, and any values of these quantities having this ratio will satisfy the system.

Exercises

Find which of the following systems have another than the zero solution, and find the solution when it exists.

1. $3x + 2y - z = 0.$

$4x + 7y - 3z = 0.$

$x - 7y + z = 0.$

2. $6x + y + 3z = 0.$

$2x + y - z = 0.$

$x - y + 4z = 0.$

3. $3x + 2y - 3z = 0.$

$5x - y + 2z = 0.$

$6x + y - 3z = 0.$

4. $2x + y - z = 0.$

$4x + 8y + z = 0.$

$2x - y - 2z = 0.$

5. $2x + 5y - z = 0.$

$3x - y + 4z = 0.$

$2x + 3y + z = 0.$

6. $3x + y + 7z = 0.$

$x - y + z = 0.$

$x + 2y + 4z = 0.$

7. $6x + 3y + 9z = 0.$

$5x - y + 4z = 0.$

$x - 2y - z = 0.$

8. $2x + y - 3z = 0.$

$5x - y + 3z = 0.$

$x - 6y - 2z = 0.$

Cumulative Review

1. (a) Prove that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is equal to $-p$ and that the product of the roots is equal to q .

(b) In the equation $ax^2 - 3x + k = 0$, what must be the value of k in order that the product of the roots shall be twice their sum?

2. What information, if any, do Descartes' Rules of Signs give in regard to the roots of the equation

$$x^3 - x - 21.3 = 0?$$

Compute the largest root of this equation correct to three significant figures.

3. Evaluate the determinant

$$\begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ -1 & 7 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ -2 & 3 & 9 & -1 & 0 \\ 5 & -3 & 6 & 4 & 8 \end{vmatrix}.$$

4. Show that $(2 + i)^2 = \frac{7 + i}{1 - i}$ when $i = \sqrt{-1}$.

5. How many distinct arrangements of the letters in the word *parallel* can be formed?

6. Solve the simultaneous equations

$$\begin{aligned} x^2 + xy + 2y^2 &= 74. \\ 2x^2 + 2xy + y^2 &= 73. \end{aligned}$$

7. (a) If $y = \frac{1}{2}(e^x - e^{-x})$, simplify $\sqrt{1 + y^2}$.

(b) Simplify $\{(a^{-2})^{-\frac{2}{3}}\}^{-\frac{3}{4}}$. $\{(a^{-2})^{-\frac{3}{4}}\}^{-\frac{2}{3}} \div (a^{-\frac{3}{4}})^{-\frac{2}{3}}$.

(c) Simplify

$$\left\{ \frac{1-u}{1+u} \right\}^{-\frac{1}{2}} [(1+u)^{\frac{3}{2}}(1-u)^{\frac{1}{2}} + (1-u)^{\frac{3}{2}}(1+u)^{\frac{1}{2}}] \\ \div \left\{ 1 + \left[\frac{\sqrt{1-u}}{\sqrt{1+u}} \right]^{-2} \right\}.$$

8. One root of the equation $3x^3 - 4x^2 + 5x + 6 = 0$ is rational. Find all the roots without using Horner's method.

9. Find the limit as x approaches 2 of $\frac{x^2 + 2x - 3}{x^2 - 2x + 3}$.

10. In the expansion of $\left(3x - \frac{1}{2x}\right)^{10}$ write down the term not containing x .

11. Develop the determinant

$$\begin{vmatrix} 0 & x & 1 & 3 \\ x & 0 & -1 & 2 \\ 1 & 3 & 0 & 4 \\ -1 & 2 & 1 & 0 \end{vmatrix}.$$

12. In a club of 50 members there are 3 foreigners. If a committee of 5 is chosen by lot, what is the chance that the committee will include all the foreigners?

13. From the relation

$$x + y : x - y = 7 : 3$$

determine the ratio of x to y .

14. If a , b , and c are in progression,

then $b^2 > ac$, if arithmetic

or $b^2 = ac$, if geometric

or $b^2 < ac$, if harmonic. (See page 399, authors' Second Course in Algebra.) Prove.

15. Factor (a) $x^7 + x$.

(b) $x^2 - a^2 + y^2 - 4 - 2xy + 4a$.

(c) $64x^{6a} - y^{12a}$.

(d) $a^4 + 4$.

(e) $(a + b)^2 - (a^2 - b^2) - 6(a - b)^2$.

(f) $a^4 - 14a^2b^2 + b^4$.

16. A rubber ball is dropped from a height of 20 feet. At each rebound, it rises to $\frac{9}{16}$ of the height from which it fell. How far does it travel before it comes to rest?

17. The area of a certain right triangle is 84 sq. ft., and the area of the square on the hypotenuse is 625 sq. ft. Find the lengths of the sides.

18. Show that the equation

$$x^3 + x + 3 = 0$$

has one negative and two imaginary roots. Compute the negative root correct to three significant figures.

19. If $a:b = b:c$, prove that $a:c = (a+b)^2:(b+c)^2$.

20. A man has a hours at his disposal. How far may he ride in a coach which travels b miles an hour, so as to be back in time, if he walks back at the rate of c miles an hour?

21. If a and b are unequal and positive, determine which is greater: $a^2 + 3b^2$, or $2b(a+b)$.

22. Solve $\frac{1}{x} + \frac{1}{y} = 5$.

$$\frac{1}{x+1} + \frac{1}{y+1} = \frac{17}{12}.$$

23. The first and seventh terms of a geometrical progression are respectively $\frac{a}{b}$ and $\frac{b^5}{a^5}$. Find the third term.

24. Find the ratio of the ninth term to the eighth term of

$$\left(1 + \frac{3x}{2}\right)^{12}.$$

25. Prove that the square of the arithmetical mean of two quantities is equal to the arithmetical mean of the arithmetical and geometrical means of the squares of the same two quantities.

26. Solve $x^2 + y^2 = 4$,
 $y^2 = x - 2$,

and illustrate the solution by the use of graphs.

27. Solve for y by determinants.

$$\begin{aligned}(a + b)x + (a - b)y + z &= 2a + 1. \\ ax + by + z &= a + b + 1. \\ x + y + z &= 3.\end{aligned}$$

28. Expand $(1 + x)^n$ to four terms. Apply to the finding of $\sqrt[3]{1.06}$.

29. Given $f(x) = x^2 - 3x + 2$; find $f(x + 4)$; $f(x + h)$.

30. (a) Locate graphically the roots of the equation

$$x^3 - 6x^2 + 9x - 2 = 0.$$

(b) Determine approximately from the graph the smallest value or values of the function $x^3 - 6x^2 + 9x - 2$ between the limits $x = 0$ and $x = 4$.

CHAPTER XI

UNDETERMINED COEFFICIENTS

THE principle explained in this chapter has a wide variety of applications, only a few of which are treated here. We are attempting only to give you an insight into an extremely useful method which may help you in your later work in advanced mathematics.

134. Principle of Undetermined Coefficients. — The fundamental theorem on which this method is based is called the *Principle of Undetermined Coefficients* and may be stated thus :

If two functions of x , each of the n th degree, are equal to each other for more than n values of x , the coefficients of like powers of x in the two functions are equal.

This theorem may be proved as follows :

$$\begin{aligned} \text{Let} \quad a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n \\ \qquad \qquad \qquad = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \dots + b_n \end{aligned}$$

for more than n values of x .

Transposing, we obtain

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + (a_2 - b_2)x^{n-2} + \dots + (a_n - b_n) = 0,$$

and this equation is true for more than n values of x .

Hence by section 90 each coefficient must be zero.

$$\text{That is, } a_0 - b_0 = 0. \qquad \qquad \text{Hence } a_0 = b_0.$$

$$a_1 - b_1 = 0. \qquad \qquad a_1 = b_1.$$

$$a_2 - b_2 = 0. \qquad \qquad a_2 = b_2.$$

$$\dots \dots \dots \qquad \qquad \dots \dots \dots$$

$$a_n - b_n = 0. \qquad \qquad a_n = b_n.$$

Clearly such an equation is an identity.

This principle has no application to functions with numerical coefficients for $2x^2 - 8x + 3 = 3x^2 + x - 1$ cannot be true for more than two values of x . (See section 90.)

The principle above stated may be applied if the two functions are not of the same degree, since the coefficients of missing terms are then considered as 0.

Thus, if $Ax^3 + Bx^2 + Cx + D = Mx^2 + Nx + P$ for more than 3 values of x , $A = 0$, $B = M$, $C = N$, and $D = P$.

The following illustrations suggest elementary applications of this principle.

Illustrative example I:

By the method of undetermined coefficients, find the quotient when $30x^4 + 11x^3 - 82x^2 - 12x + 48$ is divided by $3x^2 + 2x - 4$.

Solution: Since we are dividing a polynomial of the *fourth* degree by one of the *second* degree, the resulting quotient will be of the second degree, and hence of the form $Ax^2 + Bx + C$, where A , B , and C are *undetermined coefficients*.

We therefore assume

$$\begin{aligned} 30x^4 + 11x^3 - 82x^2 - 12x + 48 \\ \equiv (3x^2 + 2x - 4)(Ax^2 + Bx + C). \end{aligned}$$

Expanding,

$$\begin{aligned} 30x^4 + 11x^3 - 82x^2 - 12x + 48 \\ \equiv 3Ax^4 + (3B + 2A)x^3 + (3C + 2B - 4A)x^2 \\ + (2C - 4B)x - 4C. \end{aligned}$$

Since this equation is an identity,

$$3A = 30. \quad (1)$$

$$3B + 2A = 11. \quad (2)$$

$$3C + 2B - 4A = -82. \quad (3)$$

$$2C - 4B = -12. \quad (4)$$

$$-4C = 48. \quad (5)$$

$A = 10$. From (1).

$B = -3$. From (2) by substitution.

$C = -12$. From (3) by substitution.

And these values also satisfy equations (4) and (5).

Hence the resulting quotient is $10x^2 - 3x - 12$.

Illustrative example II:

By the method of undetermined coefficients, find the product of $(x - 1)(2x + 1)(3x - 5)$.

Solution: The product will be of the third degree in x .
Hence

$$(x - 1)(2x + 1)(3x - 5) \equiv Ax^3 + Bx^2 + Cx + D,$$

where the coefficients A , B , C , and D are to be determined.

Solution: Let $x = 0$, then $5 = D$.

Let $x = 1$, then $0 = A + B + C + D$.

Let $x = 2$, then $5 = 8A + 4B + 2C + D$.

Let $x = 3$, then $56 = 27A + 9B + 3C + D$.

Solving the above systems of equations, we obtain

$$A = 6, B = -13, C = 2, D = 5.$$

Hence the product is $6x^3 - 13x^2 + 2x + 5$.

Exercises

1. If $x^2 - 6x + 3 = A(4x^2 - 2) + B(3x + 6) - C(2x^2 - 4x)$, for all values of x , determine the values of A , B , and C .

2. If $10x^2 - mx + 8$ is exactly divisible by $2x - 4$, determine the value of m .

3. What is the condition that $x^2 + mx + n$ may be exactly divisible by $x - c$?

Hint: Assume $x^2 + mx + n \equiv (x - c)(x + k)$.

Then write two equations and eliminate k from them.

4. What is the condition that $x^2 + lx + m$ shall be a perfect square?

5. What is the condition that $ax^2 + bx + c$ shall be exactly divisible by $mx + n$?

Hint: Assume $ax^2 + bx + c \equiv (mx + n) \left(\frac{a}{m}x + k \right)$.

6. What is the condition that $x^3 + bx + c$ may be divisible by a perfect square factor of the form $x^2 + 2mx + m^2$?

Using the method of undetermined coefficients, find

7. The product of $(x + 3)(x - 2)(3x + 1)$.

8. The product of $(5x^2 - 8)(x + 3)(x - 6)$.

9. The quotient of $2x^3 - 9x^2 + 11x - 3$ by $2x - 3$.

10. The quotient of $2x^4 + 5x^3 - 8x^2 + 11x - 20$ by $x + 4$.

11. The quotient of $x^4 + x^2 + 1$ by $x^2 - x + 1$.

12. The quotient of $x^6 - x^4 + x^3 - x^2 + 2x - 1$ by $x^2 + x - 1$.

135. Partial Fractions. — Probably all of you could combine the fractions $\frac{3}{x-1} - \frac{5}{x+2}$ and obtain as a result

$\frac{11 - 2x}{(x-1)(x+2)}$, as the process of adding fractions is a com-

mon one. Up to this time, however, you have had no method by which the process may be reversed, and the sum

$\frac{11 - 2x}{(x-1)(x+2)}$ broken up into the two fractions of which it

is composed. These are called *Partial Fractions*. The method of doing this is shown by solving the above problem as an

Illustrative example I :

We know that the given denominator $(x - 1)(x + 2)$ is obtained by taking the lowest common multiple of the denominators of the partial fractions. Hence there can be no fraction contained in the given fraction which has a denominator other than factors of the given denominator. We assume therefore that the denominators of the partial fractions were $x - 1$ and $x + 2$. It is understood too that the partial fractions are *proper fractions*, that is, that the numerators are of lower degree than the denominators. If, now, we denote the numerators by A and B , we have the identity

$$\frac{11 - 2x}{(x - 1)(x + 2)} \equiv \frac{A}{x - 1} + \frac{B}{x + 2}.$$

Clearing of fractions, we obtain

$$11 - 2x \equiv A(x + 2) + B(x - 1).$$

In this identity we may assume $x = 1$.

Then $9 = 3A$ or $A = 3$.

If we assume $x = -2$, we obtain

$$15 = -3B \text{ or } B = -5.$$

Hence
$$\frac{11 - 2x}{(x - 1)(x + 2)} = \frac{3}{x - 1} - \frac{5}{x + 2}.$$

How could you check your result?

Illustrative example II :

Resolve into partial fractions $\frac{2x^2 - 6x - 2}{x^2 - 4x + 3}.$

Solution : Before the method of the preceding example can be applied it is necessary that the numerator shall be of lower degree than the denominator. Since that is not the case in

this example we divide the numerator by the denominator, obtaining

$$2 + \frac{2x - 8}{x^2 - 4x + 3}.$$

We now assume

$$\frac{2x - 8}{x^2 - 4x + 3} \equiv \frac{A}{x - 3} + \frac{B}{x - 1}.$$

Hence $2x - 8 = A(x - 1) + B(x - 3).$

Let $x = 1$, then $-6 = -2B$ and $B = 3.$

Let $x = 3$, then $-2 = 2A$ and $A = -1.$

Hence $\frac{2x^2 - 6x - 2}{x^2 - 4x + 3} = 2 - \frac{1}{x - 3} + \frac{3}{x - 1}.$

In case the denominator contains an unfactorable quadratic factor the method of solution is indicated in

Illustrative example III:

Resolve into partial fractions $\frac{x^2 - 7x - 5}{(x^2 + 1)(x - 2)}.$

Since it is only necessary that our resulting fractions shall have their numerators of lower degree than their respective denominators, we shall assume that the numerator of the fraction whose denominator is $x^2 + 1$ is of the *first* degree in x , and hence of the form $Ax + B$. Of course the numerators *may be* numerical. If so, we shall find $A = 0$, when we have solved our problem.

We therefore assume

$$\frac{x^2 - 7x - 5}{(x^2 + 1)(x - 2)} \equiv \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 2}.$$

Then $x^2 - 7x - 5 = (Ax + B)(x - 2) + C(x^2 + 1).$

Let $x = 2$. Then $-15 = 5C$ and $C = -3.$

Also by equating coefficients

$$A + C = 1, B - 2A = -7.$$

By substituting $C = -3$ we obtain $A = 4$ and $B = 1$.

$$\text{Hence } \frac{x^2 - 7x - 5}{(x^2 + 1)(x - 2)} = \frac{4x + 1}{x^2 + 1} - \frac{3}{x - 2}.$$

Illustrative example IV :

Resolve into partial fractions $\frac{3x^2 - x - 10}{(x - 1)^3}$.

Solution: $(x - 1)^3$ is a common denominator for fractions having as denominators $x - 1$, $(x - 1)^2$, and $(x - 1)^3$. Hence we have to assume the possibility that fractions having all these denominators are present, though of course we do not know such to be the case.

We therefore assume

$$\frac{3x^2 - x - 10}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}.$$

We assume that the numerators corresponding to the denominators $(x - 1)^2$ and $(x - 1)^3$ are numerical, for if one of them had a numerator of higher degree, the fraction could be broken up into two or more fractions with numerical numerators. For example, suppose one fraction to be

$$\frac{2x + 4}{(x - 1)^2}.$$

$$\text{Then } \frac{2x + 4}{(x - 1)^2} = \frac{2(x - 1) + 6}{(x - 1)^2} = \frac{2}{x - 1} + \frac{6}{(x - 1)^2}.$$

Hence when the given fraction is completely decomposed, the numerators will be numerical.

Clearing of fractions,

$$3x^2 - x - 10 = A(x - 1)^2 + B(x - 1) + C.$$

In this identity let $x = 1$. Then $C = -8$.

Equating coefficients of like powers of x ,

$$A = 3. \quad -2A + B = -1. \quad B = 5.$$

$$\text{Hence } \frac{3x^2 - x - 10}{(x - 1)^3} = \frac{3}{x - 1} + \frac{5}{(x - 1)^2} - \frac{8}{(x - 1)^3}.$$

Exercises

Resolve into partial fractions :

$$1. \frac{8x - 19}{(x - 2)(x - 3)}.$$

$$2. \frac{16 - 13x}{(3x - 1)(x + 2)}.$$

$$3. \frac{x + 12}{6x^2 - x - 1}.$$

$$4. \frac{12x^2 + 11x - 13}{(x - 1)(x + 1)(2x + 3)}.$$

$$5. \frac{2x - 6x^2 - 12}{(x^2 + 3)(x + 3)}.$$

$$9. \frac{18x^2 + 27x + 6}{(1 + 3x)^3}.$$

$$6. \frac{-16x^2 - 9x - 26}{(3x^2 + 5)(x - 4)}.$$

$$10. \frac{96x^2 + 144x + 64}{(4x + 3)^3}.$$

$$7. \frac{x^3 - 2x^2 + 13x}{x^4 - 1}.$$

$$11. \frac{-x^2 - 15x - 2}{(3x^2 + 1)(x - 3)}.$$

$$8. \frac{3x - 1}{(x - 2)^2}.$$

$$12. \frac{3x^3 - 12x^2 + 9x - 21}{(x^2 + 5)(x - 2)^2}.$$

136. Summation of Series.—The method of Undetermined Coefficients enables us to develop formulas for the sum of n terms in certain series whose n th terms are functions of n .

Illustrative example : Determine a formula for the sum of n terms of the series $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \dots$.

Solution : We must first determine the law of the series, so that we may express the n th term as a function of n . This is a matter of inspection and trial. We note that each term is the product of two factors. The first factor is equal to twice the number of the term less one, while the second factor is 2 greater than the first. Hence the n th term is $(2n - 1)(2n + 1)$.

It may be proved that the sum of such a series is a rational integral function of n , and that its degree is not greater than that of the n th term $+ 1$. In this case therefore we assume a function of the third degree in n .

Thus :

$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + 7 \cdot 9 + \dots \equiv A + Bn + Cn^2 + Dn^3$
where A, B, C , and D are to be determined.

Since this equation is an identity it is true for all values of n .

If $n = 1$, we have, $1 \cdot 3 = A + B + C + D$.

If $n = 2$, we have, $1 \cdot 3 + 3 \cdot 5 = A + 2B + 4C + 8D$.

If $n = 3$, we have,

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 = A + 3B + 9C + 27D.$$

If $n = 4$, we have,

$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + 7 \cdot 9 = A + 4B + 16C + 64D.$$

That is, $A + B + C + D = 3$.

$$A + 2B + 4C + 8D = 18.$$

$$A + 3B + 9C + 27D = 53.$$

$$A + 4B + 16C + 64D = 116.$$

Solution of this system gives $A = 0$, $B = -\frac{1}{3}$, $C = 2$,
 $D = \frac{4}{3}$.

Hence the sum equals

$$-\frac{1}{3}n + 2n^2 + \frac{4}{3}n^3 = \frac{n}{3}(4n^2 + 6n - 1).$$

Verify when $n = 5$ and when $n = 6$.

Exercises

Determine a formula for the sum of n terms of each of the following series :

1. $1 + 4 + 9 + 16 + \dots$

2. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots$

3. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \dots$
4. $1 \cdot 3 + 4 \cdot 6 + 9 \cdot 11 + \dots$
5. $1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \dots$
6. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots$
7. $2 \cdot 4 + 4 \cdot 8 + 6 \cdot 12 + \dots$
8. $1^3 + 2^3 + 3^3 + 4^3 + \dots$
9. $2^3 + 4^3 + 6^3 + 8^3 + \dots$
10. $2 \cdot 4 + 4 \cdot 7 + 6 \cdot 10 + \dots$

Cumulative Review

1. Solve $\frac{3.7x}{.9} - \frac{2\frac{3}{4}x}{.02} = 1.72 + \frac{x}{.04}$ and find value correct to three decimal places.
2. An athletic field is surrounded by a circular track. A bicycle cyclometer indicates 132 feet more in making the circuit on the outer edge than on the inner edge. Find the width of the track, assuming that the circumference of a circle is $3\frac{1}{2}$ times the diameter.

3. Reduce $\sqrt[29]{\left[\left(\frac{a^{\frac{3}{5}}}{a^{-\frac{5}{3}}}\right)^{-\frac{3}{5}} \div \left(\frac{a^{\frac{1}{6}} \cdot a^{-\frac{1}{3}}}{a^{-\frac{1}{5}} \cdot a^{\frac{2}{3}}}\right)^{\frac{3}{4}}\right]^{-40}}$ to its simplest form without radicals.

4. Given $(3x^2 + \frac{3}{5})(1 + 5x^2)^{-\frac{1}{2}}5x - 6x\sqrt{1 + 5x^2}$. What is its value if $x = 1.2$?

5. Given $\frac{3 + 5\sqrt{6}}{2\sqrt{3} + 3\sqrt{2}}$. Find the value to two decimals if $\sqrt{2} = 1.414$ and $\sqrt{3} = 1.732$.

6. Solve $x^2 + 1.3x + .2 = 0$ correct to three significant figures.

7. Find correct to three significant figures the negative root of the equation: $1 - \frac{2}{x+1} + \frac{4x}{(x+1)^2} = 0$.

8. Solve $15x^4 + 49x^3 - 92x^2 + 28x = 0$.

9. For what values of k does the equation

$$\frac{x}{k-1} + \frac{1}{x-1} = \frac{x}{k}$$

have (a) one root equal to zero? (b) equal roots?

10. A rectangular piece of tin is made into an open box by cutting a 5-inch square from each corner and turning up the sides and ends. If a 3-inch square were cut from each corner, the box, made in the same way, would hold the same amount. The width of the tin is 4 inches less than its length. How much does the box hold?

11. A man walked 12 miles at a certain rate and then 6 miles farther at a rate one half a mile an hour faster. Had he walked the whole distance at the faster rate, his time would have been 20 minutes less. Find his rate.

12. The diagonal of a rectangle is 89 inches. If each side of the rectangle were 3 inches shorter, the diagonal would be 85 inches. Find the length of each side.

13. A cistern has 3 pipes. To fill it, the first pipe takes one half of the time required by the second, and the second takes two thirds of the time required by the third. If the three pipes be open together, the cistern will be filled in 6 hours. In what time will each pipe fill the cistern?

14. What number can be added to 2 , $6\frac{1}{2}$, $2\frac{3}{4}$, $8\frac{3}{4}$ so that the resulting numbers will form a proportion?

15. In the expansion of $\left(3x - \frac{1}{3x^{\frac{1}{2}}}\right)^7$ find the term which, when simplified, contains $x^{\frac{5}{2}}$.

16. A man has 3 coats, 4 vests, and 5 pair of trousers. In how many ways can he dress?

17. Find by the use of determinants the values of x and y in the following system of equations:

$$\begin{aligned}\frac{x}{2} + \frac{y}{3} + \frac{z}{4} &= 7. \\ x + 2y + 3z &= 48. \\ \frac{x}{3} - \frac{2y}{3} + \frac{z}{3} &= 4.\end{aligned}$$

18. Obtain by trial two roots of the equation

$$x^4 + 4x^3 - 6x^2 + 24x - 72 = 0$$

and then find the other roots.

19. Find one root, between 0 and 1, of the equation

$$x^3 + 8.9x^2 - 0.54x - 0.80 = 0$$

correct to three significant figures.

20. If three coins are tossed, what is the probability that they will turn up two heads and one tail?

21. Evaluate
$$\begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & 1 & -2 \\ 1 & -1 & 2 & 4 \\ 2 & -2 & 4 & 6 \end{vmatrix}.$$

22. Represent graphically the sum and difference of the two complex numbers $\frac{1 + 18i}{3 + 4i}$ and $\frac{3 - 29i}{3 - 4i}$ where $i = \sqrt{-1}$.

23. Separate $\frac{5x^2 - 6}{x(x - 1)^2}$ into partial fractions.

24. In how many different ways can 7 coins be arranged in a row? In a circle?

25. Given the inequality $x - z < y$, the terms x , y , and z being positive; show the effect of multiplying the inequality by $+a$, by $-a$ if a is positive in each case; of changing the signs of all the terms; of transposing z .

CHAPTER XII

LOGARITHMS

IN solving mathematical problems much time is consumed in performing arithmetical operations which are often long and tedious. In this chapter we shall take up the use of logarithms by which the amount of time and labor involved in such operations is greatly lessened.

137. Logarithms. — *The logarithm of a number to a given base is the exponent of the power to which the base must be raised to produce the given number.*

A study of this definition shows that the emphatic words, or those which form the principal parts of the sentence, are those we have printed in italics. They state that *a logarithm is an exponent*. The definition shows too that in order to have a logarithm there must be a *base* or number to which the logarithm must be applied as an exponent.

For example, $2^3 = 8$.

Here 2 = the base.

3 = the exponent or logarithm.

8 = the resulting number.

This statement may also be written

$$\log_2 8 = 3.$$

When written in this way the word *logarithm* is abbreviated *log*, and the base used is written as a subscript following it. Hence you see that $b^x = n$, and $\log_b n = x$, mean exactly the same thing.

138. Bases Used. — John Napier, the discoverer of logarithms (see Historical Note on page 281), used, as a base, a number denoted by ϵ . This is an irrational number like π ,

and usually defined thus: $\epsilon = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$.

Its value is approximately 2.7182. While this is useful for theoretical purposes it is less convenient in practical work than a rational base. Logarithms computed to base ϵ are called *Napierian* or *natural* logarithms.

Any positive number except 1 could be the base of a system of logarithms. Explain why 1 cannot be used as a base by attempting to answer the questions: $\text{Log}_1 1 = ?$ $\text{Log}_1 2 = ?$

Why can a negative number not be used as a base?

Since 10 is the base of our number system its use as a base has certain advantages over any other, and that is the base usually used in practical work. Logarithms computed to base 10 are called *common logarithms*. When no other base is stated, the base 10 is implied.

The change from exponential form to logarithmic form will be better understood by the following

Illustrative example I :

Write $2^5 = 32$ in the corresponding logarithmic form.

Solution: By reference to the standard exponential and logarithmic forms found in the last two lines of section 137, we have $\log_2 32 = 5$.

Illustrative example II :

Write $\log_6 1296 = 4$ in exponential form.

Solution: Following the same principles as in example I,

$$6^4 = 1296.$$

Illustrative example III :

Write $\log_2 128 = x$ in exponential form and then find the value of x .

$$\begin{aligned}\text{Solution:} \qquad 2^x &= 128. \\ 2^x &= 2^7.\end{aligned}$$

Now since the bases are equal, then the exponents must be equal.

$$\text{Or} \qquad x = 7.$$

Exercises

1. Write in logarithmic form each of the following statements: $2^4 = 16$, $3^2 = 9$, $5^3 = 125$, $4^{\frac{3}{2}} = 8$, $5^0 = 1$, $3^{-1} = \frac{1}{3}$, $2^{-3} = \frac{1}{8}$, $16^{-\frac{3}{4}} = \frac{1}{8}$, $25^{-\frac{1}{2}} = \frac{1}{5}$.

2. Write in exponential form each of the following, and then find the value of x in each case.

$$\log_3 27 = x, \quad \log_2 32 = x, \quad \log_4 16 = x, \quad \log_2 256 = x, \\ \log_8 16 = x.$$

3. State the value of each of the following:

$$\log_2 64, \log_4 64, \log_8 64, \log_{16} 64, \log_3 81, \log_9 81, \log_{27} 81, \\ \log_5 25, \log_5 125, \log_{25} 125, \log_7 49, \log_7 343, \log_{49} 343, \log_9 3, \\ \log_{16} 2, \log_{125} 5, \log_{81} 27, \log_2 \frac{1}{8}, \log_5 \frac{1}{125}, \log_4 \frac{1}{2}, \log_{16} \frac{1}{8}, \\ \log_{125} \frac{1}{25}, \log_{\frac{2}{3}} \frac{3}{2}, \log_{\frac{1}{8}} 4, \log_{10} 1.$$

139. Theorems concerning Logarithms. — The following theorems state principles governing the use of logarithms to any base.

1. *The logarithm of 1 to any base is 0.*

Proof: Let a = any number.

$$\text{Then} \qquad a^0 = 1.$$

$$\therefore \log_a 1 = 0.$$

2. *The logarithm of the base itself is 1.*

Proof: $a^1 = a.$

$$\therefore \log_a a = 1.$$

3. *The logarithm of a product is equal to the sum of the logarithms, of its factors, or, stated in symbols,*

$$\text{Log}_a MN = \log_a M + \log_a N.$$

Proof: Let $x = \log_a M$, so that $a^x = M$.

Let $y = \log_a N$, so that $a^y = N$.

Multiplying these equations, $a^{x+y} = MN$.

Hence, $\log_a MN = x + y = \log_a M + \log_a N$.

4. *The logarithm of a quotient is equal to the logarithm of the dividend or numerator, minus the logarithm of the divisor or denominator, or, stated in symbols,*

$$\text{Log}_a \frac{M}{N} = \log_a M - \log_a N.$$

Proof: Let $x = \log_a M$, so that $a^x = M$.

Let $y = \log_a N$, so that $a^y = N$.

Dividing the first equation by the second, $a^{x-y} = \frac{M}{N}$.

Hence, $\log_a \frac{M}{N} = x - y = \log_a M - \log_a N$.

5. *The logarithm of a power of a number equals the logarithm of the number multiplied by the exponent of the power, or, stated in symbols,*

$$\text{Log}_a M^n = n \cdot \log_a M.$$

Proof: Let $x = \log_a M$, so that $a^x = M$.

Raising both members of this equation to the n th power,
 $(a^x)^n = a^{xn} = M^n.$

Hence, $\log_a M^n = xn = n \cdot \log_a M$.

6. *The logarithm of a root of a number is the logarithm of the number divided by the index of the root, or, stated in symbols,*

$$\text{Log}_a \sqrt[n]{M} = \frac{1}{n} \log_a M.$$

Proof: Let $x = \log_a M$, so that $a^x = M$.

Extracting the n th root of both members of the equation,

$$\sqrt[n]{a^x} = a^{\frac{x}{n}} = \sqrt[n]{M}.$$

$$\text{Hence, } \log_a \sqrt[n]{M} = \frac{x}{n} = \frac{1}{n} \log_a M.$$

You should thoroughly memorize the proofs of these important theorems, as they are the authority for the rules stated on page 274.

140. Common Logarithms. — Common logarithms have already been defined. The advantages of using base 10 rather than any other base will be pointed out in section 142. The table on pages 268 and 269 are tables of common logarithms.

141. Characteristic and Mantissa. — Since most numbers are not exact powers of 10, their logarithms to base 10 are irrational numbers. Thus, since $10^1 = 10$, $\log 10 = 1$. Since $10^2 = 100$, $\log 100 = 2$. Since 73 is a number between 10 and 100, $\log 73$ is a number between 1 and 2. As a matter of fact it is 1.8633 correct to the nearest figure in the fourth decimal place. A table like the one in our book which gives the logarithm to the nearest ten-thousandth is called a *four-place table*.

The *characteristic* of a logarithm is the part of the logarithm which precedes the decimal point. It may be either positive or negative.

The *mantissa* is the decimal part of a logarithm. It is always positive.

Thus, in the logarithm of 73, the characteristic is 1 and the mantissa is .8633.

The table of logarithms gives only mantissas, since the characteristic can be inserted by the following

Rules. — (1) *If the number whose logarithm is desired is greater than 1, the characteristic is positive, and numerically one less than the number of figures to the left of the decimal point in the given number.*

(2) *If the number whose logarithm is desired is less than 1, write it in decimal form. The characteristic is then negative and numerically one greater than the number of zeros immediately following the decimal point.*

When the characteristic is negative the $-$ sign should not be written in front of the logarithm, since it does not apply to the logarithm as a whole, but to the characteristic only. Instead we add 10 to the negative characteristic and subtract it again at the end of the logarithm. For example, the characteristic -1 should be written 9. mantissa -10 , the characteristic -2 should be written 8. mantissa -10 , etc. Example, $\log .03 = 8.4771 - 10$.

From the above rules and illustrations you will see that the characteristic depends entirely on the position of the decimal point in the given number. The mantissa, on the other hand, is entirely independent of the position of the decimal point, but depends solely on the arrangement of the figures in the given number. Thus the mantissas of the logarithms of 72,350, 7235, 72.35, .7235, and .0007235 are exactly the same, while their characteristics are 4, 3, 1, -1 , and -4 , respectively.

142. Advantage of Base 10. — The fact that we are able to determine the characteristic by inspection makes it unnecessary to print it in the tables.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

The fact that mantissas of all numbers having the same sequence of digits are the same, makes it unnecessary to have separate tables for different positions of the decimal point.

These advantages come about from the fact that the base used is the same as the base of our number system. If any other base were used, it would greatly increase the number of tables necessary.

Exercises

1. State the characteristic of the logarithm to base 10 of each of the following numbers :

82.16, 3285, 1.845, 346.8, 50436, .62, .0865, .00028, .00006, .003, 50.167, 3890.05, 8.046.

2. State the position of the decimal point in numbers whose logarithms to base 10 have the following characteristics :

3, - 1, 2, 0, 4, 1, - 2, - 3, 9 ... - 10, 8 ... - 10, 7 ... - 10, 4 ... - 10.

3. Given $\log 386.5 = 2.5872$, write

$\log 38.65$, $\log 3.865$, $\log 38650$, $\log .0003865$.

143. The Direct Use of the Tables. — The direct use of the table of logarithms consists in finding the logarithm of a given number. For example, let it be required to find the logarithm of 76.8. Since there are two figures to the left of the decimal point, the characteristic is 1. To find the mantissa we look for the first two figures of the given number in the left-hand column of the table, headed N. We find 76 near the middle of that column on the second page of the table. We then follow the horizontal line to the right until we come to the column headed 8 at the top. Here we read the mantissa .8854. Hence $\log 76.8 = 1.8854$.

144. Interpolation. — If the given number contains more than three figures, we cannot find the logarithm directly from

the table. As an illustration let us find the logarithm of 1864. The characteristic is 3. We shall, for the moment, consider the decimal point as after the third figure, thus 186.4. Doing so will not affect the mantissa since that is independent of the position of the decimal point. Now, as in the preceding example, we look up the mantissa of 186 and obtain .2695. We then look up the mantissa of 187 and obtain .2718. There is a difference between these mantissas of .0023. Now since the given number is .4 of the way between 186 and 187 we assume that the corresponding mantissa will be .4 of the way between .2695 and .2718. We therefore multiply .0023 by .4, carrying the result to the nearest figure in the fourth decimal place, and obtain .0009. This is added to the mantissa of 186 as a correction, giving the desired mantissa as .2704. Hence $\log 1864 = 3.2704$.

This process of inserting a mantissa for 186.4 between the mantissas of 186 and 187 is called *interpolation*. The method used suggests that the logarithm varies as the number. While this is not true as a general principle, it is nearly enough true over a small interval so that the above process gives the correct result.

Illustrative example I:

Find the logarithm of .93775. The characteristic is -1 , which we write in the form $9 \dots - 10$. The mantissa corresponding to the figures 937 is .9717, while that corresponding to 938 is .9722. Multiplying the difference .0005 by .75, we obtain .0004 to the nearest ten-thousandth. Adding this to .9717, we obtain .9721. Hence $\log .93775 = 9.9721 - 10$.

We now state the following

Rule. — *To find from the table the logarithm of any number*

(1) *Note the position of the decimal point, and write the proper characteristic.*

(2) *Look up in the table the mantissa of the number expressed by the first three significant digits, and that of the next larger number.*

(3) *Find the difference between these mantissas, and multiply that difference by the remaining figures of the given number preceded by a decimal point.*

(4) *Carry this result to the nearest figure in the fourth decimal place, and add it to the mantissa of the first three figures as a correction.*

145. Inverse Use of the Tables. — The inverse use of the table consists in finding the number corresponding to a given logarithm. The number which corresponds to a given logarithm is often called the *antilogarithm*.

Illustrative example I :

Let it be required to find the antilogarithm of 3.7731. The table does not contain characteristics, hence we do not look for the 3. Instead we look among the mantissas for .7731, and find it near the top of the second page of the table. The corresponding number is read by taking the two figures at the left of the page in the same row as the mantissa .7731 for the first two figures, and the figure at the top of the page in the same column as .7731 for the third figure. This gives us 593. Since the characteristic is 3, there are four figures before the decimal point in the number. Hence the desired number is 5930.

The following indicates the procedure when the mantissa cannot be found exactly in the table.

Illustrative example II :

Find the antilogarithm of 1.5744.

We cannot find this mantissa in the table, but we do find the mantissas .5740 and .5752 between which it lies. We

write the three figures corresponding to the smaller mantissa .5740, and obtain 375. The difference in the table between .5740 and the mantissa corresponding to 376, .5752, is .0012. The difference between .5740 and the mantissa we are looking for, .5744, is .0004. Now we assume that, since the given mantissa is $\frac{4}{12}$ or $\frac{1}{3}$ of the way between .5740 and .5752, the corresponding number is $\frac{4}{12}$ or $\frac{1}{3}$ of the way between 375 and 376. The work may be arranged as follows:

$$\begin{array}{r} 376 = .5752 \\ x = .5744 \\ 375 = .5740 \\ \hline \frac{4}{12} = \frac{1}{3} \end{array}$$

We change $\frac{1}{3}$ to decimal form, carrying it only to the nearest tenth, and annex the result to 375 as the fourth figure of our antilogarithm. This process is also called interpolation. The characteristic 1 shows that there are two figures before the decimal point. Hence the resulting antilogarithm is 37.53.

For finding an antilogarithm, when the given logarithm is not found exactly in the table, we state the following

Rule. — (1) *Locate the given mantissa between two consecutive mantissas in the table, and write the three figures corresponding to the smaller of them.*

(2) *Find the difference between these two consecutive mantissas in the table, and also the difference between the smaller of them and the mantissa whose antilogarithm you desire to find.*

(3) *Divide the latter difference, by the former, carrying the result to the nearest tenth.*

(4) *Annex the figure just found to the three figures already written as the fourth figure of the antilogarithm.*

(5) *Insert the decimal point by inspection of the characteristic of the given logarithm.*

Exercises

Find the logarithm of each of the following numbers :

- | | | |
|------------|------------|---------------|
| 1. 28.69. | 5. 228.62. | 9. .30465. |
| 2. 382.11. | 6. 1416.3. | 10. .0862. |
| 3. 4.8692. | 7. .8691. | 11. .004981. |
| 4. 1238.6. | 8. 1.2141. | 12. 86432000. |

Find the antilogarithm corresponding to each of the following logarithms :

- | | | |
|------------------|------------------|------------------|
| 13. 1.8162. | 17. 1.4628. | 21. 9.1287 - 10. |
| 14. 7.5670 - 10. | 18. 3.0067. | 22. 8.0632 - 10. |
| 15. .8197. | 19. 4.2792. | 23. 3.2986. |
| 16. 2.4321. | 20. 8.6283 - 10. | |

146. Operations by Use of Logarithms. — We may multiply, divide, raise to powers, and extract roots by means of logarithms, but we cannot add or subtract. The method of using logarithms in each of the above cases may be found by a study of the theorems proved in section 139.

They may be stated in the following

Rules. — (1) *To multiply two or more numbers add their logarithms and look up the antilogarithm of the result.*

(2) *To divide one number by another subtract the logarithm of the divisor from that of the dividend, and look up the antilogarithm of the result.*

(3) *To raise a number to a power, multiply the logarithm of the number by the exponent and look up the antilogarithm of the result.*

(4) *To extract a root of a number, divide the logarithm of the number by the index of the root, and look up the antilogarithm of the result.*

A negative number has no real logarithm to a positive base. Since a negative sign occurring in connection with the operations named will affect only the sign of the result, problems involving them should be worked as though the signs were all positive. After the numerical value of the result has been obtained, its sign should be inserted according to the usual laws of signs in algebra.

147. Cologarithms. — The cologarithm of a number is the logarithm of the reciprocal of that number.

$$\begin{aligned}\text{That is,} \quad \text{colog } x &= \log \frac{1}{x} \\ &= \log 1 - \log x \\ &= 0 - \log x \\ &= -\log x.\end{aligned}$$

Hence the cologarithm of a number is also the negative of the logarithm of the number. In performing the operation of division by use of logarithms we may, therefore, add the cologarithm of the divisor instead of subtracting its logarithm.

In practice, the cologarithm is obtained from the tables by subtracting the logarithm from 0 in the form of 10.0000 — 10.

$$\begin{aligned}\text{Thus} \quad & \log 24.84 = 1.3952. \\ \text{Hence} \quad & \text{colog } 24.84 = 8.6048 - 10. \\ \text{Also} \quad & \log .0868 = 8.9385 - 10. \\ \text{Hence} \quad & \text{colog } .0868 = 1.0615.\end{aligned}$$

Obviously the cologarithm is negative when the logarithm is positive and vice versa.

You will find it very easy to write the cologarithm by merely looking at the logarithm when you note that the right figure of the logarithm is subtracted from 10 and every subsequent one from 9.

The advantage of the use of cologarithms is that it enables us to arrange our work more compactly by putting it all down in a single tabulation as in the following

Illustrative example I:

Compute the value of $\frac{86.14 \times 141.23}{1.561 \times 293.2}$.

$$\begin{array}{rcl}
 \text{Solution:} & \log 86.14 & = 1.9352 \\
 & \log 141.23 & = 2.1499 \\
 & \text{colog } 1.561 & = 9.8066 - 10 \\
 & \text{colog } 293.2 & = 7.5328 - 10 \\
 & \hline
 & \log \text{ ans.} & = 1.4245 \\
 & \text{Ans.} & = 26.58.
 \end{array}$$

Illustrative example II:

Compute the value of $\frac{(8.16)^3 \cdot \sqrt[3]{167.2}}{3.921(.027)^2}$.

$$\begin{array}{rcl}
 \text{Solution:} & 3 \log 8.16 & = 2.7351 \\
 & \frac{1}{3} \log 167.2 & = .7412 \\
 & \text{colog } 3.921 & = 9.4066 - 10 \\
 & * 2 \text{ colog } .027 & = 3.1372 \\
 & \hline
 & \log \text{ ans.} & = 6.0201 \\
 & \text{Ans.} & = 1,047,000.
 \end{array}$$

148. Exponential Equations. — An exponential equation is an equation in which the unknown quantity occurs in an exponent, as $2^x = 16$ and $3.7^x = 12$. If x has a rational value, it can be determined by inspection. Thus, $x = 4$ in the first example above. In general, however, the solution of exponential equations requires the use of logarithms.

Illustrative example I:

Solve $2^x = 32$ for the value of x without the use of logarithms.

* Subtract twice the log of .027 from 20.0000 — 20.

Solution: $2^x = 32.$

$$2^x = 2^5.$$

$$\therefore x = 5. \quad \text{Since their bases are the same.}$$

Illustrative example II:

Solve $3^{-x} = 9$ for the value of x , without logarithms.

Solution: $3^{-x} = 9.$

Then $3^x = \frac{1}{9}.$

$$3^x = \frac{1}{3^2}.$$

$$3^x = 3^{-2}.$$

$$\therefore x = -2.$$

Illustrative example III:

Express in the language of exponents, $\log_3 9 = x$, and find the value of x .

Solution: Recalling that in your study of logarithms you had to deal with three numbers, (1) a number, (2) its logarithm, (3) the base used, and keeping in mind the fact that $\log_B N = L$, may also be written

$$B^L = N.$$

We then have $3^x = 9.$

Or, $3^x = 3^2.$

$$\therefore x = 2.$$

Illustrative example IV:

Solve $3.7^x = .609.$

Solution:

Take the logarithm of both members of the equation.

Then, $x \log 3.7 = \log .609.$

$$.5682 x = 9.7846 - 10 = -.2154$$

$$x = \frac{-.2154}{.5682} = -.379^+$$

Illustrative example V:Solve $17^{x-1} = (4.75)^{2x+3}$.*Solution:*

$$(x - 1) \log 17 = (2x + 3) \log 4.75.$$

$$(x - 1) 1.2304 = (2x + 3) .6767.$$

$$1.2304x - 1.2304 = 1.3534x + 2.0301.$$

$$- .1230x = 3.2605.$$

$$x = - 26.508.$$

After the numerical value of the logarithm has been inserted in an example like the one just worked, it becomes an ordinary linear equation such as you found in your first-year algebra. Success requires care in properly placing the decimal point.

Exercises

The first 10 examples are to be worked without the use of the tables.

1. Given $\log 2 = .30103$, $\log 3 = .47712$ and $\log 10 = 1\ 00000$, find $\log 6$, $\log 12$, $\log 27$, $\log 32$, $\log 5$, $\log \frac{3}{2}$, $\log \sqrt{2}$, $\log \sqrt[3]{30}$, $\log 33\frac{1}{3}$, $\log 720$, $\log 125$.

2. How many figures are there in 32^{40} ?

3. How many zeros before the first significant digit in $(.0672)^{20}$? ($\log 7 = .8451$.)

4. Show that $(\frac{1}{15})^{36}$ is greater than 10.

5. Express $\log \sqrt{s(s-a)(s-b)(s-c)}$ in terms of the logs of its factors.

6. Show that $2 \log \frac{9}{8} + 3 \log \frac{2}{3} + \log 16 - \frac{1}{2} \log 9 = \log 2$.

7. What is the value of $6^{\log_6 10}$? Why?

8. Solve

- (a) $2^x = 16$. (c) $3^{x+1} = 27$. (e) $8^x = 16$. (g) $2^x = 1\frac{1}{28}$.
 (b) $3^x = 81$. (d) $5^{-x} = 125$. (f) $4^x = 32$. (h) $(\frac{3}{2})^x = \frac{4}{9}$.

9. Express in the language of logarithms :

(a) $3^4 = 81$. (b) $2^7 = 128$. (c) $3^{-2} = \frac{1}{9}$. (d) $4^{\frac{3}{2}} = 8$.

10. Find the base of the system of logarithms in which

(a) $\log 27 = 3$. (c) $\log \frac{1}{16} = -4$.

(b) $\log 81 = \frac{4}{3}$. (d) $\log \frac{1}{27} = -\frac{3}{2}$.

Compute by logarithms the value of

11. $(36.42)(111.42)(3.46)^2(.0176)$.

12. $\left(\frac{8.643}{1.21}\right)^2$.

14. $\frac{\sqrt[3]{-811.2} \cdot \sqrt{161.7}}{-3.758}$.

13. $\frac{(.063)^3 \cdot (814.6)}{2.266}$.

15. $\frac{2.763(-.473)^2}{\sqrt{104.7} \cdot (3.21)^5}$.

16. $\frac{(861.42)\sqrt[6]{21.477}}{\sqrt[5]{-121.4}}$.

17. $(3.087)^4 \cdot (1.234)^2 \div (-3.5421)^3$.

Solve

18. $(3.7)^x = 111.42$.

19. $(2.14)^x = 62.875$.

20. $(3.86)^{x+1} = .4863$.

Hint: The negative characteristic must be combined with the positive mantissa in the logarithm of .4863 to form an ordinary negative number, before division is possible.

21. $.3^x = .8$.

22. $62.4^{2x-3} = 1.0861$.

23. $3.75^x = .0472$.

24. $21.43^{x-1} = 48.06^{2x+3}$.

25. $6^x = 3^{2x} \cdot 2.417^{x-1}$.

26. $2^{x^2-6x} = (\frac{1}{2})^{x+1}$. Solve without logarithms.

27. The area of a $\odot$ is given by the formula $A = \pi r^2$.

Find the area of a $\odot$ whose radius is 24.861. ($\pi = 3.1416$.)

28. The volume of a sphere may be computed by the formula $V = \frac{4}{3} \pi r^3$. If $\pi = \frac{22}{7}$, and $r = 6.281$, find V . Compute by logarithms.

29. The volume of a cylinder is found by the formula $V = \pi r^2 h$. If $\pi = \frac{22}{7}$, $r = 4.712$, and $h = 10.96$, compute the value of V by logarithms.

30. A bushel contains 2150.42 cubic inches. Find the number of cubic feet in a bin which will exactly contain 82.76 bushels.

31. Find the number of cords of wood in a pile 32.6 feet long, 12.72 feet wide, and 8.36 feet high. (A cord contains 128 cubic feet.)

32. If $A = p(1 + r)^n$, where n is the number of years and p is the principal, find the amount of \$8575 at 4% compound interest, compounded annually, at the end of 10 years.

33. What annual pension will \$10,000 buy for 20 years if money is worth 4%?

$$\text{Hint: An. Pension} = Pr \cdot \frac{(1 + r)^n}{(1 + r)^n - 1},$$

$$\text{where } P = \$10,000, n = 20, r = .04.$$

34. The formula for the amount at the end of n years of \$ p placed on interest at $r\%$, and compounded annually is $A = p(1 + r)^n$. In how many years will any amount, say \$100, double itself at 6% interest compounded annually?

35. How many years will it take \$1 to double itself at 4% compound interest compounded annually?

$$\text{Hint: } 1.04^x = 2.$$

36. How many years will it take \$100 to double itself at $4\frac{1}{2}\%$ compound interest, compounded annually?

37. Substitute $\frac{r}{2}$ for r and $2n$ for n in the formula in example 34, and find how many years it will take \$100 to double itself at 4% compound interest compounded semi-annually.

38. What is the present value of an annual pension of \$1200, if money is worth 4%, and the pension is to run for 18 years?

$$\text{Pres. Val.} = \frac{A}{r} \left(1 - \frac{1}{(1 + r)^n} \right).$$

39. The population of New York State in 1910 was 9,113,614 and in 1920 it was 10,385,227. What was the average yearly rate of increase if Av. Yearly Rate = $R - 1$

and $R = \sqrt[n]{\frac{P_e}{P_b}}$, where P_e is the population at the end of the period and P_b is the population at the beginning of the period?

HISTORICAL NOTE

The discovery of the method of logarithms is due to Baron John Napier, a Scotch nobleman, who was born in Merchiston in 1550 and died in 1617. He was deeply interested in political and religious matters and wrote extensively on these topics. He pursued the study of mathematics as a recreation.

Nearly all his discoveries in mathematics were of a kind intended to save labor, and to make computations easier, and complicated processes more simple. Logarithms have been called the greatest labor-saving device ever discovered.

We should also remember that while to-day we may think of logarithms in connection with exponential equations, you should also know that exponents were not known to Napier nor were they understood for a century and a half after his time, when Euler noted the relation between exponents and logarithms.

Napier did not use the base 10, which we use to-day. He published a work concerning his discoveries in 1614, and soon thereafter Henry Briggs, a professor in Oxford University, England, having read this treatise, made a special trip to Scotland to meet Napier.

It is said of this meeting that when Briggs came into the presence of Napier, they greeted each other, and then almost a quarter of an hour elapsed while each looked at the other and neither spoke a word. Finally Briggs spoke as follows: "My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this excellent help in astronomy, viz., logarithms; but, my lord, being by you found out, I wonder why nobody found it out before, when now known it is so easy."

CHAPTER XIII

MAXIMA AND MINIMA

IN the authors' Second Course in Algebra we discussed briefly the maximum and minimum points of a curve. We again in Chapter IX of this book continue a discussion of the function graph, section 88, and the graphic solution of quadratic equations, sections 96 to 99. The subject of maximum and minimum points of a curve again came up in the latter discussion in this book.

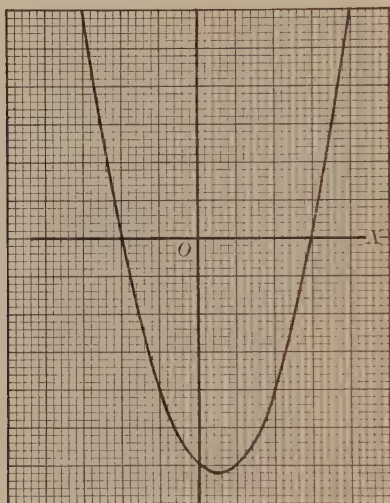
We now propose to enlarge upon this topic and give you more definite and conclusive work along this line as a preparation for your study of higher mathematics.

149. Graphic Treatment of Maxima and Minima. — Let us consider a rational, integral function of x , as for example $f(x) = x^2 - x - 6$. In order that we may study the way in which this function changes as the value of x changes, we shall draw its graph (see fig. 1) as explained in section 88. The following table gives values of x and the corresponding values of $f(x)$.

x	0	1	2	3	4	- 1	- 2	- 3
$f(x)$	- 6	- 6	- 4	0	6	- 4	0	6

You should clearly understand that the resulting parabola is a diagram which shows the changes in $f(x)$ as x changes from $- 3$ to $+ 4$. The *ordinate* of any point is the value of $f(x)$ when x has the value denoted by the *abscissa* of that point. Thus, the abscissas of points where the graph crosses the x -axis are values of x for which $f(x)$ is 0. That is, they are values of x which satisfy the equation $x^2 - x - 6 = 0$.

These give us the roots 3 and -2 . Values of x between 3 and -2 make $f(x)$ negative, while values of x which are less than -2 or greater than 3 make $f(x)$ positive.



Graph of the quadratic function
 $x^2 - x - 6$

We now desire to find the value of x which gives the *least* or *minimum* values of $f(x)$, and also to determine what that minimum value is.

Generally speaking, if we assume $f(x)$ to have some definite value, say 10, and set $f(x)$ or $x^2 - x - 6 = 10$, we shall find by solving the equation that there are two values of x which will make $f(x)$ have this value.

There is only one value of x , however, which will make the function assume its minimum value. If now we denote $f(x)$ by y , and write

$$x^2 - x - 6 = y$$

or

$$x^2 - x - (6 + y) = 0,$$

we must find that value of y for which the roots of the equation are equal, and this will give us the minimum value of the function. Hence we have the following

$$\text{Solution:} \quad x^2 - x - (6 + y) = 0.$$

This equation will have equal roots if

$$b^2 - 4ac = 0. \quad (\text{See section 80.})$$

$$\text{Here } a = 1, b = -1, c = -(6 + y).$$

$$\text{Hence } 1 + 4(6 + y) = 0.$$

$$\text{From which } 1 + 24 + 4y = 0,$$

$$\text{or } 4y = -25$$

$$\text{and } y = -6\frac{1}{4}.$$

By substituting this value of y in $y = x^2 - x - 6$, we find that the corresponding value of x is $\frac{1}{2}$.

By closely observing the graph and using paper ruled either 10 or 20 spaces to the inch you will be able to check by sight the fact that $y = -6\frac{1}{4}$ and $x = \frac{1}{2}$ is the minimum value of the function or the lowest point on the curve.

You will also note that there is but one value of y for which the values of x are equal on this curve.

By taking the equation

$$ax^2 + bx + c = y,$$

where a is positive, we may find in a similar manner that the coördinates of the minimum point of the curve are

$$y = -\frac{b^2 - 4ac}{4a}.$$

$$x = -\frac{b}{2a}.$$

You will observe that if the value of y in the above for the minimum point of the curve is positive, the equation has no real roots, for the curve will not touch the x -axis.

If a is negative, the form of the graph will remain the same but will be inverted. In this case we shall be able to find by

the above equations the highest or *maximum* point of the curve.

The graphs of the functions will show high points or low points corresponding respectively to maximum or minimum values of the function.

A *maximum* value of a function is a value which is greater than all values immediately preceding or following.

A *minimum* value of a function is a value which is less than all values immediately preceding or following.

Exercises

Graph the following equations, find the points where the graphs cut the x -axis. Find in each, the lowest or highest point of the curve.

1. $x^2 + x - 6 = 0$.

4. $2x^2 + 5x - 3 = 0$.

2. $x^2 - 2x - 8 = 0$.

5. $3x^2 - 2x - 2 = 0$.

3. $x^2 - 4x + 3 = 0$.

6. $x^2 - 2x + 1 = 0$.

By inspection of the graph, estimate the maximum and minimum points of the following.

7. Using the values in order from -3 to $+3$, form a set of values for $y = 2x^3 - 3x^2 - 12x + 12$, and then plot the graph of the function. What value of x gives the maximum (or largest) possible value for the function?

8. What value of x will make the function x^2 have the least possible (or minimum) value?

9. Find the maximum value of $2 - 3x - x^2$ when x is real.

10. Given the formula $d = 100t - 16t^2$, which represents that a ball has been thrown upwards with an initial velocity of 100 feet per second and where d represents the distance that it will rise. Construct the graph of the function and find the maximum value.

11. Separate 10 into two parts such that their product shall be a maximum.

Hint: What represents the parts? What represents their product? Their product is the function required.

12. Separate 10 into two parts such that the sum of their squares shall be a minimum.

13. Plot $y = x^3 - 3x^2 - 9x + 5$ between $x = -2$ and $x = 4$, and from your graph determine a maximum and a minimum value of the curve.

14. Find the number of square feet of ground in the largest rectangular field that can be inclosed by 800 feet of fence.

Find the maximum and minimum values of each of the following :

15. $1 + 4x - x^2$.

16. $2x^2 - x + 4$.

17. Find the maximum or minimum value of $y = x^2 + 6x - 7$.

18. A maker of cardboard boxes wishes to make the largest possible open box out of cardboard 6 inches by 4 inches by cutting out small squares at the corner and turning up the edges, thus forming the box. We wish to find the side of the small square.

Hint: Let x equal the side of the square to be cut.

The length of the box will be $6 - 2x$.

The width of the box will be $4 - 2x$.

The depth of the box will be x .

$$\begin{aligned}\text{Hence the volume } V &= x(6 - 2x)(4 - 2x) \\ &= 24x - 20x^2 + 4x^3,\end{aligned}$$

or

$$\frac{V}{4} = 6x - 5x^2 + x^3.$$

Using suggested values of $0, \frac{1}{2}, 1, \frac{3}{2}, 2$, for x , form a table of values for the function and from these draw a graph. Note the points where the graph changes direction and tell what you think should be the value for x in order that he may construct the box.

19. Work a similar problem where the cardboard is 8 inches by 12 inches.

20. Construct a circle whose diameter is 4 inches. Find the sides of the rectangle having the largest area which may be inscribed within the circle.

Hint: Let the length of a base be x . Now the diameter of this circle (which is also the diagonal of the rectangle) is 4, and from this we find that the altitude of our rectangle is $(16 - x^2)^{\frac{1}{2}}$.

If A represents the area in square inches, we obtain

$$A = x(16 - x^2)^{\frac{1}{2}}.$$

Here we have the function A and its related variable x . Let the x values be plotted as usual for x values, each unit representing 1 inch.

Let the A values be plotted as ordinates, and each unit representing 1 sq. in.

If carefully plotted on good graph paper ruled 10 or 20 to the inch, you will be able to read the ordinate of the maximum point quite accurately. This ordinate gives the largest possible area, while the corresponding abscissa gives the length of one side of the rectangle which has that area.

21. In a similar manner find the length of the sides of the largest rectangle which may be inscribed in a circle with a diameter of 10 inches. Of 5 inches.

What is your conclusion about the nature of the largest rectangle which may be inscribed in a circle?

22. The perimeter of a rectangle is 40 yards. What is the length and the width if the area is to be a maximum?

23. What number exceeds its square by the greatest amount?

24. Rectangles are to be inscribed in a circle whose radius is 2. Graph the perimeter P of the rectangles as a function of x . For which rectangle is P a maximum?

25. Plot each of the following equations and study their graphs for the minimum value in each case.

(a) $y = x^2$.

(c) $y = 3x^2$.

(b) $y = 2x^2$.

(d) $y = \frac{1}{2}x^2$.

26. Construct a graph of the equation

$$y = x^3 - 3x + 2$$

and find out from your graph what you can concerning maximum or minimum points. What is your conclusion?

27. Plot a graph of the function

$$x^2 + 2x - 4,$$

and from your graph determine

(a) The minimum value of the function.

(b) The values where the function crosses the x -axis.

28. Construct a graph of the equation

$$A = x(6 - x),$$

and from your graph determine a maximum value of the function

$$x(6 - x).$$

150. Maxima and Minima by Use of Quadratics. — We noted in section 149 that it was possible to determine maximum and minimum values of a function by the use of the graph. We shall now show you how it is possible to find these values by the quadratic method, because the latter is more accurate and often not as long as the former method.

It is entirely possible that as x increases, y (or the function) will increase to a certain value which we shall call m , and then decrease, or that y will decrease to a certain value which we will call m' , and then increase.

You are to understand that m is to represent the *maximum* value and m' the *minimum* value of the function y .

For example, $y = (x - 1)^2 - 6$ has a minimum value when $x = 1$. This value is -6 . If x begins at a value less than 1 and increases, then $(x - 1)^2$ will decrease to 0 and then increase.

Similarly $y = 6 - (x - 1)^2$ will have a maximum value, 6, when $x = 1$.

We may find a maximum or a minimum value of any quadratic trinomial $ax^2 + bx + c$ (having real coefficients) as shown in the following

Illustrative example I :

Find the maximum or minimum value of

$$y = x^2 - 4x - 10.$$

Solution : By completing the square we have

$$x^2 - 4x - 10 = (x - 2)^2 - 14.$$

Hence if $x = +2$, y has a minimum value of -14 .

Illustrative example II :

Separate a given line 6 inches long into two parts, which shall be the sides of a rectangle with the greatest possible area.

Solution : Let x represent the length of one part in inches. Then $6 - x$ will represent the length of the other part in inches.

Let y represent the area of the desired rectangle.

Then $y = x(6 - x) = 6x - x^2.$

Or, $y = 9 - (3 - x)^2 = 3^2 - (3 - x)^2.$

Therefore y will have a maximum value when $x = 3$; in other words, when the given line is bisected and therefore our rectangle is a square whose area is 3^2 and each side is 3 inches.

Illustrative example III:

Having considered in the last two examples certain special quadratics, we shall now consider the general quadratic function $ax^2 + bx + c$, where it is understood that all coefficients are integral and a is positive.

Solution:

$$y = ax^2 + bx + c.$$

$$\therefore y = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$$

Hence by completing the square

$$y = a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right].$$

The above form shows that y has a minimum value when $x = -\frac{b}{2a}$. In that case the value of y is

$$a\left(\frac{4ac - b^2}{4a^2}\right) = \frac{4ac - b^2}{4a}.$$

As x increases from $-\infty$ to $+\infty$, y or $f(x)$ will first decrease from $+\infty$ to $\frac{4ac - b^2}{4a}$, which value it reaches when x reaches $-\frac{b}{2a}$. From this point the function increases again to $+\infty$.

Exercises

Find the maximum or minimum value of each of the following functions:

1. $x^2 - 6x + 8$.
2. $3x^2 - 4x - 1$.
3. $20x - 2x^2$.

4. $-x^2 - x - 1$. 5. $2 - 3x - x^2$. 6. $2x^2 - x + 4$.
 7. Find the maximum or minimum value of $y = x^2 + 6x - 7$.

8. Divide 10 into two parts such that their product shall be a maximum.

9. Divide 10 into two parts such that the sum of their squares shall be a minimum.

10. Find the maximum or the minimum values of $1 + 4x - x^2$.

11. Consider the values that are possible for the expression $\frac{x^2 - x + 1}{x^2 + x + 1}$, when x is real.

Solution: Let $\frac{x^2 - x + 1}{x^2 + x + 1} = y$.

Then $x^2 - x + 1 = y(x^2 + x + 1)$.

Or, $x^2(1 - y) - x(1 + y) + 1 - y = 0$.

If x is real, we now must have

$$(1 + y)^2 - 4(1 - y)^2 > 0. \quad \text{That is,} \quad b^2 - 4ac > 0.$$

Or $-3(y - \frac{1}{3})(y - 3) > 0$.

$$\therefore y > 3 \text{ or } y < \frac{1}{3}.$$

Hence also no real value of x that will make $y = 0$, since $\frac{1}{3} \leq y \leq 3$.

12. Show algebraically that $\frac{x^2 - 6x + 5}{x^2 + 2x + 1}$ cannot be less than $-\frac{1}{3}$ if x is real. Check graphically.

13. Obtain the maximum or minimum points of $y = -x^2 - x + 3$.

14. Determine the maximum or minimum values of the following functions.

(a) $x^2 - 7x + 12$.

(b) $x^2 + 4 + 4y = 4x$.

(c) $x^2 - 8x + 3$.

15. Find the sides of a rectangle having the greatest area which may be inscribed in a circle whose diameter is 4 inches. Also the greatest in a circle whose radius is 5 inches.

Hint: Since the area is positive, find the sides of the rectangle for which the square of the area is a maximum.

16. What height will a body reach if thrown vertically upward from the ground with an initial velocity of 100 feet per second, and when will it reach that height?

Hint: $h = 100t - 16t^2$.

17. Find the number of square rods of ground in the largest rectangular field that can be inclosed by 80 rods of fence.

18. What number exceeds its square by the greatest amount?

151. Maxima and Minima by Use of the Derivative. — We shall attempt to give you a little notion of how the derivative may be applied to finding maximum and minimum values, but this subject must be treated very briefly and superficially as you will need to know more mathematics such as is found in the Calculus if you are to enjoy the full advantages and exactness of its applications.

In sections 101 and 102 you learned something about the derivative and maxima and minima. We shall now extend this type of work in order that you may use it to find maximum and minimum values.

Your graphs of functions and algebraic solutions by use of the quadratic have taught you how to find the high points or the low points of certain curves which correspond to maximum or minimum values of the function.

In section 149 we defined maximum and minimum values of a function.

One condition for a maximum or a minimum value is that the tangent to the curve at such a point shall be parallel to the x -axis. In other words the slope of the curve at that point shall be zero.

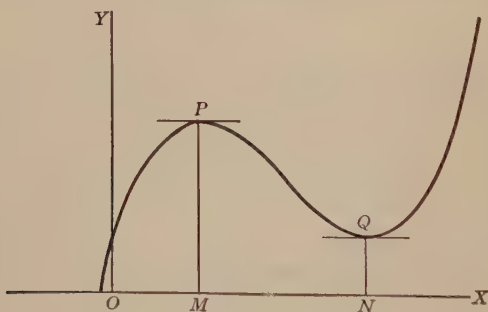
For an explanation of slope of a line see authors' Second Course in Algebra, page 449.

In our discussion of this topic we shall use the following notation :

$f'(x)$ represents the first derivative of the function y .

$f''(x)$ represents the second derivative of the function y , which means that it is the first derivative of the first derivative of that function.

In the following graph you will observe that at both P and Q the tangent to the curve is parallel to the x -axis.



You will also note that as we move along this curve from left to right, at P the slope changes from positive to negative ; but at Q it again changes from negative to positive.

Or

At P the slope *decreases* as x increases.

At Q the slope *increases* as x increases.

It may be proved in the Calculus that when $f'(x) = 0$, and $f''(x) < 0$, there is a *maximum* value of y (1), and when $f'(x) = 0$, and $f''(x) > 0$, there is a *minimum* value of y (2).

Illustrative example :

Find the maximum and minimum values of the function

$$2x^3 - 3x^2 - 12x + 12.$$

Solution: Let $y = 2x^3 - 3x^2 - 12x + 12$ be the given function $f(x)$.

Then $f'(x) = 6x^2 - 6x - 12$ (3). See rule section 101.

And $f''(x) = 12x - 6$ (4). Apply the rule again to (3).

Now put (3) = 0.

$$6x^2 - 6x - 12 = 0.$$

Whence $x = 2$ or -1 .

Substituting these values of x in (4), we note that

When $x = 2$, $f''(x) = 18 > 0$.

When $x = -1$, $f''(x) = -18 < 0$.

Hence by (1) and (2)

When $x = 2$, y has a minimum value.

When $x = -1$, y has a maximum value.

Now by substituting these values in the given function we find that the maximum value of y is $y = 19$ and the minimum value of y is $y = -8$. This leads us to the following

Rule. — (1) Find the first derivative of the given function.

(2) Set the first derivative equal to zero and solve the resulting equation for real roots in order to find values (called critical values) of the variable.

(3) Find the second derivative of the function.

(4) Substitute each critical value for the variable in the second derivative. If the result is negative, then the function is a maximum for that critical value; if the result is positive, the function is a minimum.

152. Another Condition Denoting Maximum and Minimum. — Referring again to the diagram in section 151, we note that the function has a maximum value when the function changes from an increasing to a decreasing function as at P .

$f'(x)$ is positive when $f(x)$ is an increasing function and negative when $f(x)$ is a decreasing function. That is, the slope of the function is positive before and until it reaches P when it then becomes negative to Q . Therefore $f'(x)$ changes from $+$ to $-$ as the function passes through P . Again $f'(x)$ changes from $-$ to $+$ as the function passes through Q .

As the derivative $f'(x)$ can only change sign as it passes through zero or infinity, therefore it follows that a value for x which makes $f(x)$ a maximum or a minimum must satisfy one of the two equations:

$$f'(x) = 0 \text{ or } f'(x) = \infty.$$

Naturally the required values are roots of these equations.

We shall consider here only the former case or the one in which our required value of x shall be a root of the equation $f'(x) = 0$.

Illustrative example I:

Find the altitude of the cylinder of the greatest volume that can be inscribed in a given segment of a paraboloid of revolution (a figure formed by the rotation of a parabola about its axis).

Solution: Let a represent this altitude and b represent the radius of the base of the segment. The equation of the parabola which will generate this paraboloid is

$$y^2 = 4cx. \quad (1)$$

Now (a, b) is a point on the curve, so by substituting these values in (1) we have

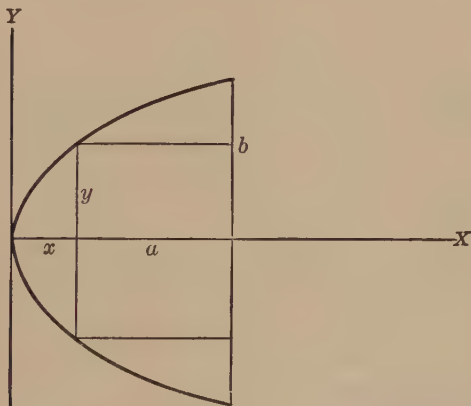
$$b^2 = 4ca. \quad (2)$$

By solving equations (1) and (2) as simultaneous equations and eliminating $4c$, we obtain as the equation of the curve

$$y^2 = \frac{b^2}{a} x. \quad (3)$$

The volume of the required cylinder is

$$V = \pi y^2(a - x).$$



By substituting from equation (3) for y^2 , we obtain

$$V = \pi \frac{b^2}{a} x(a - x).$$

Now since π , and $\frac{b^2}{a}$ are not affected by finding the derivative, we put $f(x) = x(a - x) = ax - x^2$.

Then $f'(x) = a - 2x$. Apply section 101 to each term.

We now set this value equal to zero and we have

$$a - 2x = 0$$

or

$$x = \frac{a}{2},$$

which gives us the required condition of the problem and $a - x$, which is the required altitude of the cylinder is $\frac{a}{2}$ or one half the altitude of the segment.

Illustrative example II :

Find maximum and minimum points of

$$6y = 2x^3 - 3x^2 - 12x - 6.$$

Solution: By letting $f(x)$ represent y and then by taking the derivative of every term of each side of the above equation we have

$$6[f'(x)] = 6x^2 - 6x - 12$$

or

$$\begin{aligned} f'(x) &= x^2 - x - 2 \\ &= (x - 2)(x + 1). \end{aligned}$$

Here we observe that if x has any value less than -1 , both factors of $f'(x)$ are negative, and hence $f'(x)$ is positive. The curve then rises to the right at all points for which $x < -1$.

If x is greater than -1 but less than 2 , one factor of $f'(x)$ is positive and the other negative, and therefore $f'(x)$ is negative. Hence the curve falls to the right for all values of x between -1 and 2 .

If $x > 2$, $f'(x)$ is positive and therefore the curve again rises to the right for all values of $x > 2$.

As x passes through -1 to the right, $f'(x)$ changes from positive to negative, and therefore the point of the curve for which $x = -1$ is a maximum.

By substituting $x = -1$ in $f(x)$, we obtain $f(x)$ or $y = \frac{1}{6}$. Therefore $x = -1$ and $y = \frac{1}{6}$ represents a maximum point of the given function.

As x passes through 2 to the right, $f'(x)$ changes from negative to positive and in a similar manner we find that $x = 2$ and $y = -4\frac{1}{3}$ represents a minimum point of the given function.

Exercises

1. Find the maximum value of the function $32x - x^4$.
Find the maximum and minimum values of the functions
2. $2x^3 - 11x^2 + 12x + 10$.

3. $2x^3 - 9x^2 + 36x - 50.$

4. $-x^3 - 3x^2 + 9x + 15.$

Find the maximum and minimum points of the curves represented by the following equations:

5. $6y = 2x^3 - 3x^2 - 36x - 12.$

6. $3y = x^3 - 12x + 6.$

7. $10y = 2x^3 + 9x^2 - 24x + 20.$

Find maximum and minimum values of the functions

8. $\frac{x^3}{3} - 2x^2 + 3x + 1.$

9. $\frac{x^3}{3} - x^2 + 2.$

10. Divide 10 into two such parts that the product of the square of one and the cube of the other may be the greatest possible.

Hints: Let x and $10 - x$ be the parts. Now $x^2(10 - x)^3$ is to be a maximum.

If $y = uv,$

$$f'(uv) = v(f'(u)) + u(f'(v)) \text{ and } f'[(u)^n] = nu^{n-1}(f'(u)).$$

11. A square piece of cardboard whose side is a has a small square cut out of each corner. Find the side of this square that the remainder may form a box of maximum contents.

Hints: Let x = side of the small square. The contents of the box then will be $(a - 2x)^2x.$

Let $y = (a - 2x)^2x$ and find the maximum.

12. A rectangular piece of tin 15 inches long and 8 inches wide has a square cut out at each corner. Find the side of this square in order that the remainder shall form a box of maximum contents.

13. Find the greatest right cylinder that can be inscribed in a given right cone.

Hints: Construct the figure. Let a = altitude of cone, and b = radius of base of cone. Let x = radius of base of cylinder and y = altitude of cylinder. Obtain relations from two similar triangles. Also for the volume of the cylinder. Then obtain a maximum desired.

14. An open metal tank has a square base, vertical sides, and is to hold 16 cubic feet of water. Find the length of a side of the base and the altitude if the total inside surface is to be a minimum.

Hint: Let a = alt. in feet.

x = length of a side of the square base.

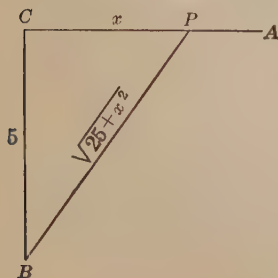
$$\text{Vol.} = 16 \text{ cu. ft.} = ax^2. \quad (1)$$

$$\text{Surface} = x^2 + 4ax. \quad (2)$$

From (1) and (2) find the surface S in terms of x , and find the minimum value.

15. The perimeter of a rectangle is 40 feet. What is the length and width of the rectangle if its area is to be a maximum?

16. A man in a boat 5 miles from the nearest point of a beach (which is straight) wants to reach in a minimum time a place 5 miles from that point along the shore. If he can run 6 miles an hour, but can row only at a rate of 4 miles an hour, find where he must land.



Hint: Let the adjoining triangle represent the direction of his trip. Then $BP = \sqrt{25 + x^2}$ and $AP = 5 - x$.

$$\frac{\sqrt{25 + x^2}}{4} = \text{number of hours of rowing.}$$

$$\frac{5 - x}{6} = \text{number of hours of walking.}$$

$$\frac{\sqrt{25 + x^2}}{4} + \frac{5 - x}{6} = \text{total time which is to be a minimum.}$$

Solve by constructing a graph of this function and the value of x will then be obtained.

CHAPTER XIV

STATISTICS

IN your future work in the world you often will be called upon to use figures or masses or groups of figures. It is very useful to know how to classify, arrange, and graph such material in order that you may represent the mass of facts in a simple way so that they shall be easily understood. It is the purpose of this chapter to give you some essential ideas about handling this type of material.

153. Meaning and Use of Statistics. — In the running of a business, a city, or a government, one is often called upon to collect and study many forms of data.

For example: A business will collect data regarding its sales from month to month and from year to year; a manufacturer will collect data on the wages paid in a similar establishment as well as the total cost of producing a similar article; a city will collect facts regarding the cost of education in other cities; the U. S. Government collects information on the cost and amount of crop production, the number of people employed in certain industries, etc., etc.

You have already studied some forms of statistics in the authors' algebras, Chapter XII of each course. In those chapters you learned something about how facts were tabulated and also how to graph such data.

154. Importance of Material and Arrangement. — In the work in statistics, the matter of arrangement of the data is highly essential, for it is from the proper arrangement of the

material that we may draw proper conclusions. It is also of great value to know what facts are needed and what facts should be omitted and how they should be obtained. The following hints will prove useful to you in your study of business statistics.

The business facts should be definite and reliable quantities that may be compared. Among these will be data appertaining to population, transportation, labor, capital, building, markets, and production.

They should be properly classified and so tabulated that they will readily and easily show proper relations and comparisons. They should be simple and as briefly stated as possible.

The chief aim should be to give a mental picture of a large mass of facts in a simplified and comprehensible tabulation. The tabulations should not only be simple, but they should be so arranged that we may obtain dependable and logical conclusions from the data thus arranged rather than from opinions or personal feelings in relation to the problem. In other words we must so arrange our material that it will convince those who are to study it.

The reliability of the statistical facts will depend to a great extent upon

- (1) The sources from which they are obtained.
- (2) The methods of obtaining them, depending upon
 - a. Personal investigation.
 - b. Estimates by outsiders.
 - c. Nature of the questionnaire. It should also provide proper checks upon the work.
- (3) Accuracy of figures reported.

The matter of the accuracy of the figures will depend upon the nature of the investigation. For example, if the study is to provide for real accuracy, we must use care with the

figures and especially in respect to decimal-place accuracy. We cannot use 7.7 and call it accurate, for the result might have come out anywhere from 7.65 to 7.74 and still our accuracy to one decimal place would be represented by 7.7. That is, 7.65 millions of dollars would obviously be quite different from 7.74 millions of dollars.

Similarly one must use great care when numbers containing decimals are multiplied, divided, raised to powers, or the root extracted. In multiplication, for example, we may safely say that our product in respect to the number of accurate decimal places will be governed by the number of decimal places in the factor having the least number of decimal places.

Thus we see that we must exercise great care if our data are to be reliable and accurate.

155. Graphs. — In the chapters of the authors' First and Second Courses in algebra above mentioned some attention was given to graphing different business facts. Such charts should be so constructed that the proper relations are brought out. They should also be accompanied with the tabulations from which they were made. In your construction of circle charts, a protractor with a percentage scale as well as a degree scale will be found useful.

156. Summarization. — When you are ready to summarize from your data or your chart you should give particular attention to

1. Your conclusions from it.
2. Your chart or tabulation appertaining to average.
 - (a) In the case of profits you must study
 - (1) The mark up.
Ex. Whether it is say 25% or 50%, etc.

(2) The number of turnovers per year. For example whether a store is turning over its sales once a month, once in three months, or once in a longer period. That is, a turnover of \$1000 @ 10% profit each month means a yearly profit of 120% on the \$1000 worth of goods sold, while a turnover of the same \$1000 once in six months means only 20% profit for the year at the same rate.

(b) You must take account of the fact that averages are often unreal. For example, the average weekly wage of nine girls receiving respectively \$23, \$24, \$24.50, \$24.40, \$10, \$15, \$9.60, \$26, and \$20 per week is \$19.61 and yet no one receives the average wage.

3. The *median* is the middle number in a series of numbers arranged in order of size. That is, if we have n numbers and n is odd, then the median number will be the $\frac{n+1}{2}$ -th number

from the top or from the bottom. If n is even, the median number will be halfway between the two middle numbers.

Ex. 1. The median weekly wage of the girls mentioned above is found upon arrangement of the wages as follows: \$26, \$24.50, \$24.40, \$24, \$23, \$20, \$15, \$10, \$9.60 to be \$23 or the middle number.

Ex. 2. Find the median if the contributions to a certain fund are \$100, \$90, \$70, \$60, \$50, \$40, \$30, \$25.

Solution: Here we have eight numbers, therefore the median is halfway between the two middle numbers \$50 and \$60 or \$55.

Ex. 3. Of two classes of 15 members each the following tabulation represents the number of examples right in a certain time test. To find the median number of rights in each class.

CLASS 1		CLASS 2	
Names by Number	Rights	Names by Number	Rights
1	21	1	20
2	18	2	17
3	17	3	16
4	15	4	14
5	14	5	13
6	13	6	12
7	12	7	11
8	11	8	10
9	11	9	10
10	11	10	10
11	10	11	9
12	10	12	9
13	9	13	8
14	9	14	8
15	6	15	7

Here you will note that if you count down from the top to the middle or the eighth number, that 11 are right in class 1 and 10 are right in class 2. We, therefore, say that 11 rights is the *median* for class 1 and 10 is the *median* for class 2. However, if you were to total the rights in class 1 and divide by 15, you would find that the average number of rights would be $12\frac{7}{15}$.

4. The mode.

The modal number represents the most common thing.

Examples.

- (a) The most common sized foot.
- (b) The most common sized head.

- (c) The most common sized man.
- (d) The most common automobile.
- (e) The most common weekly wage in a factory.
- (f) The most common daily sales or daily profits in a certain chain store.

Example. Suppose that we were to find the most common weekly wage from the following wages in a factory where

1 man receives	\$100 per week,
2 men receive	75 per week,
5 men receive	60 per week,
6 men receive	50 per week,
50 men receive	40 per week,
30 men receive	35 per week,
10 men receive	25 per week,
8 girls receive	20 per week,
2 girls receive	15 per week.

Solution: By observing the above tabulation it is easy to see that the modal sized weekly wage is \$40 because *more* people receive that wage than any other wage.

157. Central Tendencies. — The median, average, and mode are called measures of a *central tendency*.

From such examples and tabulations as you observed in the last section you will note that a given mass of facts must be arranged so that any particular measure may easily be computed from that arrangement.

You should remember that the mode, median, or average is always a *function* of the purpose that you have in mind in your problem.

The average is often used when it does not represent the central frequency as well as the mode or the median. In the above example if we arrange the data as follows :

NUMBER OF WORKERS	WAGES PER WEEK	TOTAL
1	\$100	\$100
2	75	150
5	60	300
6	50	300
50	40	2000
30	35	1050
10	25	250
8	20	160
2	15	30
114		\$4340

we find that the average $\$4340 \div 114$ is \$38.07, and no one is receiving the average wage.

The mode gives the best idea of a central tendency when there are great extremes at the top or at the bottom. The average probably shows up the central tendency in a fairly good manner when there are no great extremes, but fails in the case of finding the average sized family of say a town. The median also shows us the central tendency when there are extremes, provided the data are such that they can be arranged in an order to show the middle number.

158. The Frequency Polygon is used to represent the facts from a frequency table. It is the actual area covered by the different bars of the graph.

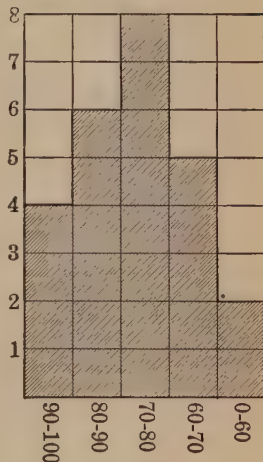
It shows quite accurately how the data are distributed. You will observe that there is a gradual rise and fall of the numbers from the lowest to the highest one. Some naturally will fall near the smallest number and some near the highest, but as a general rule a large proportion of them will be near the central tendencies.

Illustrative example I :

The following is an illustration of a frequency table.

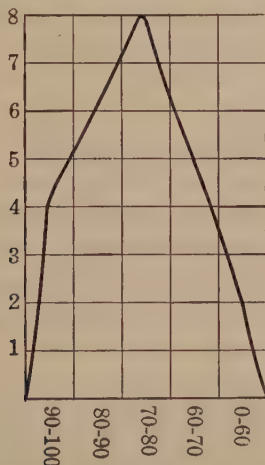
Construct a frequency polygon if a class of 25 in a certain examination made the following marks.

NUMBER OF PUPILS	PER CENT
4	90-100
6	80-90
8	70-80
5	60-70
2	0-60



Solution :

If, however, we were to graph all of the marks of the members of the class, it would probably be more nearly like the following. Here you will observe that the *mode corresponds*



to the *maximum value* of a continuous function.

By studying figure 1 we observe what the central tendency of the grades were in that examination, that is, the number of students falling within a given percentage group determine the *size* of that group or its *frequency*.

Illustrative example II :

The wealth of a certain small community is represented by the following table :

NUMBER OF PERSONS	WEALTH PER PERSON
1	\$30,000
2	15,000
5	8,000
10	5,000
14	3,000
30	500
20	200
10	0

Construct a frequency polygon from the above facts.
This exercise is left for the student.

159. Index Numbers. — Index numbers are percentages or relative numbers in which the facts for a certain year or an average of those facts for a certain year or for several years are used as a *base*, and usually they are denoted by 100.

The corresponding facts of future years may be computed in percentages upon that base. These comparisons often show *trends* which are of much value in different lines of business.

For example, \$1 might be used to show what it would buy or the number of hours' wages it would pay for in the year 1913, and from this we may show what one dollar would buy of the same kind in each of the succeeding years.

160. The Geometric Mean. — The geometric mean of n quantities $a, b, c, \dots$ is the n th root of the product of those quantities. Its advantage over the simple average is that it diminishes the influence of large numbers and increases the influence of small numbers. If it differs much from the arithmetic, it should be used, otherwise it is better to use the latter.

161. Comparisons. — In order to obtain proper comparisons you will note that you must give much attention to how you will arrange your material. In fact this is one of the most difficult parts of statistical work. You must take account of the average, the mode or the median whichever you think will best fit the problem at hand.

The data must be as reliable as it is possible to make them, and lastly your computations must have as high a degree of accuracy as the problem demands with efficient checks upon your work.

162. Deviations. — The spread or deviation of a group of numbers means the difference in their values. If the spread is small, we mean that this difference is slight as compared with the average.

For example, if 5 men applied to become members of a certain police force and their heights were respectively 6 ft., 6 ft. 1 in., 6 ft. $1\frac{1}{2}$ in., 6 ft. $1\frac{3}{4}$ in., 6 ft. 2 in., we would say that the *deviation* from the average height was very small. We then may measure this spread by the deviation from the average, mode or median. This may be expressed algebraically as follows.

Given n items $i_1, i_2, i_3, i_4 \dots i_n$ and their respective deviations from the given type (*i.e.* average, median or mode).

Let δ represent the average deviation.

Let a represent the arithmetic average.

Let M represent the median, m the mode, Σ the sum of.

Then the average deviation from the arithmetic average would be represented by

$$\delta_a = \frac{\Sigma(i - a)}{n} = \frac{\Sigma d_a}{n},$$

where $(i - a)$ or $(a - i)$ represents the deviation of any item from the average, and Σd represents the sum of all of the deviations of all of the items.

The average deviation from the mode is

$$\delta_m = \frac{\Sigma(i - m)}{n} = \frac{\Sigma d_m}{n},$$

where $(i - m)$ or $(m - i)$ represents the deviation of any item from the modal number of those items.

The average deviation from the median is

$$\delta_M = \frac{\Sigma(i - M)}{n} = \frac{\Sigma d_M}{n},$$

where $(i - M)$ or $(M - i)$ represents the deviation of any item from the median of all of the items.

Illustrative example:

Find the average deviation from the median:

FREQUENCY f	SIZE OF ITEMS i	DEVIATION FROM MEDIAN d_M	TOTAL OF fd_M
2	\$25,000	\$24,600	\$49,200
3	20,000	19,600	58,800
5	7,000	6,600	33,000
11	5,000	4,600	50,600
15	3,000	2,600	39,000
25	400	0	0
20	300	100	2,000
10	0	400	4,000
91			\$236,600

Solution:

The median is the 46th item; when all items are arranged in order of size it is found to be \$400.

Therefore the deviation from the median or $\delta_M = \frac{\Sigma d_M}{n}$
 $= \frac{236600}{91} = 2600$. This example is useful in showing the distribution of wealth in a community.

163. Coefficients of Correlation and of Dispersion. — A *correlation coefficient* is a fraction which provides us with a method by which we may, by observing a quantity Q' upon which Q depends, predict quite accurately how much of a quantity Q may be expected.

It is fairly well illustrated by the study of a child as to its height at any age. Suppose, for example, that a man's height is 5 feet 10½ inches and that of his thirteen-year-old son is 5 feet 4 inches. Let us see what our fraction will be. If we divide 70.5 (the height of the man in inches) into 52 (the height of the son), we obtain .737+. Had we made similar observations, say at each birthday of that son, by the aid of the various fractions or decimals we probably could make a fairly accurate prediction as to about what height the son would reach when he grew to manhood. Similarly we might correlate the relation between sunshine and wheat production, or the direction of the winds on different days and the amount of rainfall during that season.

The *coefficient of dispersion* is a factor by which we are enabled to weigh the different deviations according to their size. By its use the extreme deviations are given more weight than the small deviations.

Illustrative example :

Find the coefficient of dispersion from the following :

FREQUENCY f	SIZE OF FAMILY i	DEVIATION FROM MEDIAN d_M	fd_M
3	4	3	9
2	5	2	4
4	6	1	4
7	7	0	0
5	8	1	5
3	9	2	6
2	10	3	6
26	$M = 7$		38

Solution :

$$\delta_M = \frac{38}{26} = 1.461$$

$$C = \frac{\delta_M}{M} = \frac{1.461}{7} = 0.208+.$$

The student might collect similar data from your town or neighborhood and make up an example of this type.

The coefficient of dispersion for computation work gives us the following advantages :

1. Every item is considered.
2. Deviations are weighted as to their size. The greater deviations are given more weight than the smaller ones.

164. The Standard Deviation and Coefficient. — If you knew the exact height of fifty men of a certain class in West Point, you could then compute the deviation from the average height. The standard deviation is computed from the arithmetic average by the formula

$$S.D. = \sqrt{\frac{\Sigma(1 - a)^2}{n}}$$

as shown in the example on the next page:

FREQUENCY f	SIZE OF ITEM i	if	DEVIATION d	d^2	fd^2
3	7	21	- 3	9	27
2	8	16	- 2	4	8
5	9	45	- 1	1	5
7	10	70	0	0	0
8	11	88	+ 1	1	8
5	12	60	+ 2	4	20
30		300 Av. = 10			$\Sigma d^2 = 68$

$S.d.$ = Standard deviation.

$$S.d. = \sqrt{\frac{68}{30}} = 1.125^+.$$

$C.d.$ = Coefficient of dispersion.

$$C.d. = \frac{S.d.}{a} = \frac{1.125^+}{10} = 0.1125^+.$$

165. A Shortcut Method of Finding the Standard Deviation has been found and is of much value owing to the great amount of work required when the median or average is a mixed decimal or a whole number and a fraction. It is developed by observing the facts and then choosing an assumed average (which shall be a whole number) very close to the real average. The following is the

Rule. — (1) *Choose some whole number near the arithmetic average.*

(2) *Find the different deviations from this number.*

(3) *Square each of these deviations.*

(4) *Find the sum of the results of number (3).*

(5) *From this result take n times the square of the difference between the chosen number and the real average.*

(6) *Divide by n .*

(7) *Find the square root of the result in (6).*

The standard coefficient of dispersion then equals the result obtained in the above rule divided by the real average.

The above method is obtained from the following theorem.

The sum of the squares of the deviations from the arithmetic average is a minimum.

The proof of this theorem is beyond the scope of this book.

166. Uses of Deviations and Coefficients of Dispersion. —

The above topics which give a brief treatment of a part of the work in statistics are used in many studies made by economists and biologists.

In many of these studies it will be found sufficient to use only the average deviation.

For a complete treatment of this subject the student should make an extensive study of the following books:

King's "Elements of Statistical Method."

Bowley's "Elements of Statistics."

Riegel's "Elements of Business Statistics."

Secrist's "Statistics in Business."

Reitz's book on statistics.

Karsten's "Charts and Graphs."

Brinton's "Graphic Methods for Presenting Facts."

167. Errors and Approximations. — No matter how careful you are in your work it is often almost impossible not to make errors; for example, in the measurement of a block of wood or some similar thing you may not be able to get the marked part of the scale exactly even with the edge of the wood or you may not be able to read accurately with your eye the exact measurement.

It is due to these uncertainties that we must often work on as close approximations as it is possible to make. In this topic we shall attempt to show some of the examples of errors and approximations and their use to the student of statistics.

Illustrative example I :

In a measurement which gives 35.60 the zero is a significant figure, for it shows that our result is more nearly correct than 35.59 or 35.61.

Illustrative example II :

Suppose that we were to measure a piece of paper and find that the dimensions were 8.45 and 10.8 and we were to find the area.

Obviously our answer is 8.45 times 10.8. If we multiply this out, we obtain 91.26.

If our measurement were exact this would represent the correct area, but as stated above our measurement may easily be a little off from these dimensions. We can approximate our result by saying 8.5 times 11 equals 93.5. We, therefore, see that our result must be about right if our measurements are right.

Illustrative example III :

Find the area of a circle whose measured radius appears to be 2.435.

Solution :

Since our measured radius gives us four significant figures we shall use but four significant figures for π , i.e. 3.142.

$$\text{Area} = \pi r^2 = 3.142 \times 2.435 \times 2.435.$$

$2.435^2 = 5.929225$. We shall drop the last three figures and retain four significant figures.

$$3.142 \times 5.929 = 18.628918.$$

By approximation we may use 3 for π , 2 for one of the other factors and 3 for the other, obtaining 18 as our result, which shows that we are approximately correct.

Illustrative example IV :

Suppose that we want to approximate the value of the irrational number x when $x = \sqrt{3}$.

Solution:

By squaring successive integers we find that

$$1^2 < 3 \text{ and } 2^2 > 3.$$

Therefore our result must lie between 1 and 2.

Now by trying the same plan on 1.2, etc., to and through 1.8 we find that $(1.7)^2 < 3$ and $(1.8)^2 > 3$.

Therefore we conclude that our result must be $1.7 +$ something. By a similar plan we find that $(1.73)^2 < 3$ and that $(1.74)^2 > 3$. We, therefore, say that our result is $1.73 +$ etc., as far as we wish to proceed in a similar manner.

Or if we prefer to start out with 1.73 and 1.74 as our values and then test them, we find that $\sqrt{3} - 1.73 < .002$ and that $1.74 - \sqrt{3} < .008$, so we may say that whichever number we use, that number is not more than 8 thousandths out of the way.

Illustrative example V:

Find the approximate result of $\sqrt{2} \times \sqrt{5}$.

Solution:

$$\sqrt{2} \times \sqrt{5} = \sqrt{10} = 3+.$$

Illustrative example VI:

We wish to divide 36 by 2.5 and check our work by approximating:

$$36 \div 2.5 = 14.4.$$

Now we wish to approximate our result as to its accuracy. If we divide 36 by 3, we obtain 12 and if we divide 36 by 2, we obtain 18, and as our result falls between these results we can safely say that we are approximately right.

Exercises

1. Mr. Brown was injured by a truck belonging to the Smith Co. He sued the latter for \$50,000 damages. At the end of the trial the jurors could not agree upon the

amount of damages to be awarded to Mr. Brown, so they agreed to each write upon separate slips of paper the amount that each thought that Mr. Brown should receive and then strike an average from those amounts. One juror felt that Mr. Brown was entitled to \$50,000, while the others handed in amounts as follows: \$25,000; \$20,000; \$5,000; \$6,000; \$8,000; \$4,000; \$10,000; \$8,000; \$9,000; \$12,000; \$14,000. Find the average amount of damages given in the verdict by the jury.

Suppose that they had agreed that the median amount should be the proper amount, what would it have been?

Suppose that one juror had a grudge against the Smith Co., which method would seem the fairer? Suppose that one juror had a feeling that Mr. Brown was to blame and should not receive anything, which then would seem the fairer?

2. In an eighth-grade final examination in English the marks ranked as follows:

5 passed 90 to 100.
12 passed 80 to 90.
17 passed 70 to 80.
6 failed below 60.

What is the modal per cent?

About how many would you say might get on well in Latin?

3. Construct a frequency polygon from the facts in example 2.

4. Suppose that we were to take the purchasing power of one dollar in 1913 as an index or base and that you found that you could buy 10 yards of a certain material for \$1 in 1913 but could only buy 5 yards of the same material for the \$1 seven years later, what would you say the buying

power of the dollar was in the latter year as compared with 1913?

5. Find the average and then the average deviation from your arithmetic average from the following numbers: 12, 15, 17, 9, 18, 25, 11, 7, 14.

6. Change the last number under frequency in the tabulation in section 164 from 5 to 6 and then use the method of section 165 and find the standard deviation.

7. A girl measured a rectangular piece of paper and found it to be 10.65 by 12.5. Find the area and approximate your result as to correctness.

8. If I use 1.41 or 1.42 for the value of $\sqrt{2}$ in a certain computation, by how much is either of these out of the way of accuracy to the nearest thousandth?

CHAPTER XV

SHORT ADDITIONAL TOPICS

WHILE the preceding chapters contain all that is usually included in a short course in advanced algebra, there are certain topics which are occasionally needed. This concluding chapter contains a short discussion of such topics.

Indeterminate Equations

168. The Indeterminate Equation. — In your earlier courses in algebra you have learned that a single equation in two unknown quantities is called *indeterminate* because there are an indefinite number of sets of values which will satisfy it.

For example, consider the equation $3x + 4y = 18$.

In this equation we may substitute for y any value whatever and obtain a corresponding value for x , and these values constitute a set of roots of the equation. Clearly the number of such sets is unlimited. In working problems, however, the letters often represent quantities which, by their very nature, must be positive integers. In that event the number of solutions meeting these requirements is greatly limited. If we include zero among the positive integers, there are only two solutions of the above example meeting the requirements. The following is the

Solution: $3x + 4y = 18$.

$$\therefore x = \frac{18 - 4y}{3} = 6 - y - \frac{y}{3}.$$

Since x is an integer, $6 - y - \frac{y}{3}$ must be an integer, and therefore $\frac{y}{3}$ is an integer. That is, y must be a multiple of 3.

$$\begin{array}{ll} \text{If } y = 0, & \text{If } y = 3, \\ x = 6. & x = 2. \end{array}$$

If $y = 6$, the value of x is negative, and therefore rejected and the same result is obtained for values of y greater than 6. Hence the above are the only solutions in positive integers.

Illustrative example I:

Solve in positive integers.

Solution: $5x + 11y = 82.$

$$x = \frac{82 - 11y}{5} = 16 - 2y + \frac{2 - y}{5}.$$

Since x is an integer, $\frac{2 - y}{5}$ must be an integer.

Denote it by k . Then $\frac{2 - y}{5} = k$. Hence $y = 2 - 5k$ where k is any integer.

$$\begin{array}{l} \text{If } k = 0, y = 2; \\ \text{if } k = -1, y = 7; \\ \text{if } k = -2, y = 12; \\ \text{etc., etc.,} \end{array}$$

all positive values of k make y negative, and these are inadmissible.

$$\begin{array}{ll} \text{If } y = 2, & \text{If } y = 7, \\ x = 12. & x = 1. \end{array}$$

And these are the only solutions, since $y = 12$ gives a negative value of x .

Illustrative example II :

A man trebled the number of sheep in his pasture and afterwards sold 4, and in their place added 3 to the number of cows in the same pasture. The number of animals in the pasture was then 15. How many of each kind of animals were in the pasture at first?

Solution :

Let x = number of sheep in the pasture.

Let y = number of cows in the pasture.

Then $3x - 4 + y + 3 = 15$.

That is, $3x + y = 16$.

Or $x = 5 + \frac{1 - y}{3}$.

Since $\frac{1 - y}{3}$ must be an integer, we denote it by k .

Then $\frac{1 - y}{3} = k$ or $y = 1 - 3k$.

If $k = 0, -1, -2, -3$, etc., $y = 1, 4, 7, 10$, etc.

If $y = 1$, If $y = 4$, If $y = 7$, If $y = 10$, If $y = 13$,
 $x = 5$. $x = 4$. $x = 3$. $x = 2$. $x = 1$.

But the last set of answers is impossible. Why?

Illustrative example III :

Solve in least positive integers $3x - 2y = 10$.

Solution : $2y = 3x - 10$

$$y = x - 5 + \frac{x}{2}.$$

Hence $\frac{x}{2}$ is an integer, and x is a multiple of 2.

If $x < 4$, the value of y is negative.

If $x = 4$, $y = 1$, and this is the solution in *least* positive integers.

Exercises

Solve the following equations, restricting the solution to positive integers :

1. $4x + y = 12.$

3. $\frac{1}{2}x + \frac{2}{3}y = 9.$

2. $2x + 3y = 15.$

4. $\frac{3}{2}x + \frac{2}{3}y = 10.$

Find the smallest solution in positive integers of

5. $2x - 3y = 15.$

7. $7x - 2y = 32.$

6. $5x - 4y = 20.$

8. $4x - 9y = 28.$

9. Solve by the graphic method, $2x - 3y = 12.$

See authors' Second Course in Algebra, page 316.

10. Find two numbers, which when multiplied by 3 and 11 respectively have a sum of 87.

11. Divide 47 into two parts, one of which is divisible by 3, the other by 7.

12. Find the smallest number which when divided by 5 and by 8 gives each time a remainder of 3.

13. Find the smallest number which may be divided by 3 giving a remainder of 1, the quotient divided by 3 giving a remainder of 1, and the second quotient divided by 3 giving a remainder of 1.

14. A man purchased hens and turkeys for \$49.50. The hens cost 75 cents each, and the turkeys \$2.25 each. What is the least number of hens there could have been?

15. If I have only quarters and dimes, in how many ways can I pay a bill of \$12.65?

16. In how many ways can you weigh 500 pounds if you have only 7-pound weights and 12-pound weights?

Scales of Notation

169. The Decimal Scale. — In our usual method of writing numbers, 10 is the *base* or *radix* of the system. This means that we use ten characters: namely, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and that one of these characters standing in second position from the right has 10 times its face value, in third position 100 times its face value, etc. For example, the number $3472 = 3(10)^3 + 4(10)^2 + 7(10) + 2$.

170. Other Scales of Notation. — The choice of ten as a radix was an accident. Any positive integer greater than 1 might have been chosen, and it is generally conceded that 12 would have been a better base than 10. This is called the duodecimal scale. A scale with the radix 6 is called the *senary* scale, with radix 8 the *octary* scale, etc.

171. Writing Numbers in Other Scales. — Suppose that it is desired to write the number 472 in the senary scale. We must remember that we have only 6 characters; namely, 0, 1, 2, 3, 4, 5. Further, the number when written has the form $a(6^3) + b(6^2) + c(6) + d$. We can obtain d by dividing the given number by 6 and taking the remainder. We can obtain c by again dividing by 6 and taking the remainder, etc.

Thus,
$$\begin{array}{r} 6 \overline{)472} \\ \underline{6 \overline{)78} - 4} \\ 6 \overline{)13} - 0 \\ \underline{ 2} - 1 \end{array}$$
 Hence we have the number in the form 2104_6 . That is, $2104_6 = 472_{10}$.

Illustrative example I:

Write 1281 in the octary scale.

Solution:
$$\begin{array}{r} 8 \overline{)1281} \\ 8 \overline{)160} - 1 \\ 8 \overline{)20} - 0 \\ \underline{ 2} - 4 \end{array}$$
 Hence $2401_8 = 1281_{10}$.

Illustrative example II :

Write 6325 in the duodecimal scale. Here we need twelve characters, and we shall invent symbols for ten and eleven. Let * denote 10, and let # denote 11.

$$\begin{array}{r} \text{Then} \qquad \qquad \qquad 12 \overline{)6325} \\ \qquad \qquad \qquad \qquad \qquad 12 \overline{)527-1} \text{ Hence } 37\#_{12} = 6325_{10}. \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad 12 \overline{)43-\#} \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad 3-7 \end{array}$$

172. Interpreting Numbers Written in Other Scales. —**Illustrative example I :**

Write in the decimal scale 423_6 .

$$\text{Solution:} \qquad 423_6 = 4(6)^2 + 2(6) + 3 = 159.$$

Illustrative example II :

Write in the decimal scale 8203_9 .

$$\text{Solution:} \quad 8203_9 = 8(9)^3 + 2(9)^2 + 0(9) + 3 = 5997.$$

Illustrative example III :

Write in the decimal scale $6^*51\#_{12}$.

Solution :

$$6^*51\#_{12} = 6(12)^4 + 10(12)^3 + 5(12)^2 + 1(12) + 11 = 142,439.$$

Exercises

1. Write 562 in the senary scale.
2. Write in the nonary scale (radix 9) 3402.
3. Write in the quaternary scale (radix 4) 3011.
4. Write 6175 in the octary scale.
5. What number in the duodecimal scale corresponds to 2170?

Write in the decimal scale :

- | | | |
|-------------------------|----------------------------|---------------------------|
| 6. 21375 ₈ . | 9. 81*02 ₁₁ . | 12. 122102 ₃ . |
| 7. 1121 ₄ . | 10. 63# ₁₂ . | 13. 21635 ₇ . |
| 8. 5326 ₇ . | 11. 1011101 ₂ . | 14. 112401 ₅ . |

173. Reading Numbers Written in Other Scales. —

Clearly we have no choice but to read a number as it appears when written in the decimal scale regardless of the scale in which it actually is written. The resulting awkwardness results from the fact that the *names* of our numbers are made to fit the decimal scale. If we had chosen to use some other scale in our ordinary numerical work, we should have had names for the numbers which would fit the scale used.

174. Operations in Others than the Decimal Scale. —

Fundamental operations are performed on numbers written in others than the decimal scale by the same principles as those with which you are familiar. Note the following :

Illustrative example I :

Add 4126, 3275, 1063, 4214 when the radix is 8.

<i>Solution :</i>	4126	<i>Explanation :</i>	We add the right-hand
	3275		column and obtain 18 which in the
	1063		octary scale is written 22 (for there are
	4214		2 eights and 2 over in 18), hence we put
	<u>14722</u>		down 2 and carry 2 to the next column.

The second column gives the same result and is similarly treated. Addition of the third column gives 7 (since there are no eights in 7 there is 7 to put down and none to carry). Addition of the fourth column gives 12 (this is 1 eight and 4 left over), which in the octary scale is 14. Hence the result.

Illustrative example II :

Multiply 4132_6 by 123_6 .

Solution: 4132 3 times 2 = $6_9 = 10_6$. Write 0,
 123 carry 1.
 20440 3 times 3 = $9 + 1 = 10 = 14_6$, write
 12304 4, carry 1 (one six in 10 and 4 over).
 4132 3 times 1 = 3. $3 + 1 = 4$.
 1001120 3 times 4 = $12 = 20_6$ (for there
 are 2 sixes in 12 and 0 left over). In
 like manner multiply by 2 and by 1 and then add the partial
 products as in example 1.

Illustrative example III :

Subtract 14023_5 from 41121_5 .

Solution: 41121 Explanation: In order to subtract,
 14023 we must borrow 5 from the second or
 22043 (fives) column. Hence 3 from 6 is 3.
 So continue.

Illustrative example IV :

Divide 44718 by 347 , the radix being 9.

Solution: $347)44718(125$
 347
 1001
 705
 1858
 1858

Explanation: In multiplying 347 by 2 we first obtain 14 (which equals 1 nine and 5 left over). 2 times the 4 equals 8 and the 1 to carry makes 9 (this is 1 nine and 0 left over). 2 times 3 = 6. 6 and the 1 to carry makes 7, etc., etc.; then subtract as in example 3 above.

Exercises

1. Add 101201, 110011, and 122201, $r = 3$.
2. Add 5693#, 42861, 593#1, $r = 12$.
3. Subtract 4136 from 5073, $r = 8$.
4. Subtract 63281 from 150453, $r = 9$.
5. Multiply 4236_7 by 5102_7 .
6. Multiply 2963 by 4768, $r = 11$.
7. Divide 104212 by 145, $r = 7$.
8. Verify example 5 by division.
9. Verify example 6 by division.
10. Reduce to lowest terms $\frac{332}{1504}$, $r = 6$.
11. In what scale is 413 denoted by 635?
Hint: Let $r =$ radix. Then $6r^2 + 3r + 5 = 413$.
12. In what scale is 4070 denoted by 30502?
13. Show that if $r \leq 5$, 441 represents a perfect square.
Hint: $441 = 4r^2 + 4r + 1 = (2r + 1)^2$.
14. Show that 1331 is a perfect cube if $r \leq 4$.
15. Add in the octary scale

$$\frac{3}{5} + \frac{2}{3} + \frac{1}{6} + \frac{3}{4}.$$

175. Mathematical Induction proves a thing by proceeding from the particular to the general. This was illustrated in the proof of fractional exponents. It would also be illustrated in the complete proof of the Binomial Theorem of section 76. It may be further shown by a few examples.

Illustrative example I:

Prove by mathematical induction that the sum of the first n natural numbers is $\frac{n}{2}(n + 1)$.

Solution:

$$\begin{aligned}
 1 + 2 &= 3 = \frac{1}{2} \cdot 2(2 + 1) \\
 1 + 2 + 3 &= 6 = \frac{1}{2} \cdot 3(3 + 1) \\
 1 + 2 + 3 + 4 &= 10 = \frac{1}{2} \cdot 4(4 + 1) \\
 1 + 2 + 3 + 4 + 5 &= 15 = \frac{1}{2} \cdot 5(5 + 1) \\
 1 + 2 + 3 + 4 + 5 + 6 &= 21 = \frac{1}{2} \cdot 6(6 + 1)
 \end{aligned}$$

Thus far the formula seems to hold good. In other words the sum of the first n whole numbers appears to conform with the above law. Now let us try the next consecutive integer, viz., $n + 1$. We obtain

$$\begin{aligned}
 1 + 2 + 3 + 4 + 5 + \dots + n + (n + 1) &= \frac{n + 1}{2} (1 + n + 1) \\
 &= \frac{1}{2} (n + 1) (n + 2) \\
 &= \frac{1}{2} (n + 1) [(n + 1) + 1].
 \end{aligned}$$

This is of the same form as that of the law; therefore, we conclude that the law not only holds good for n terms but also for $(n + 1)$ terms, and we then conclude that the law holds for any number of terms. This gives us the following

Rule for proving by mathematical induction:

- (1) *Prove that the law holds good when $n = 1$, or 2, or 3, etc.*
- (2) *Prove that if the law holds good for any particular value of n as in (1) then that it also holds good when $(n + 1)$ is substituted for n .*
- (3) *We now conclude that the law holds good for any value whatsoever that we may assign to n .*

Illustrative example II:

We wish to prove by mathematical induction that the sum of n terms of the sequence

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}.$$

$$\text{Proof:} \quad \text{If } n = 2, \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

and

$$\frac{n}{n+1} = \frac{2}{2+12} = \frac{2}{3}.$$

Now let us assume that it does hold when we use n terms and try to prove it when we replace n with $(n+1)$.

We shall do this by adding the sum of n terms, viz., $\frac{n}{n+1}$, to the next term and we have $\frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$. By addition we actually find that this equals $\frac{n+1}{n+2}$.

Hence we observe that if the sum of n terms is $\frac{n}{n+1}$, that it follows that the sum of $(n+1)$ terms is $\frac{n+1}{n+2}$.

In the third step we found that the sum of 2 terms was $\frac{2}{3}$.

If we were to find the sum of 3 terms, it would be $\frac{3}{4}$.

Thus we observe that $\frac{n}{n+1}$ holds good for any sequence.

Illustrative example III:

Prove by mathematical induction that $a^n - b^n$ is always exactly divisible by $a - b$.

$$\text{Proof:} \quad \begin{array}{l} a^n - b^n \\ a^n - a^{n-1}b \\ \hline a^{n-1}b - b^n \end{array} \left| \begin{array}{l} a - b \\ a^{n-1} \end{array} \right. \\ \hline a^{n-1}b - b^n = b(a^{n-1} - b^{n-1}).$$

Now if $a^{n-1} - b^{n-1}$ is exactly divisible by $a - b$, then our division will come out without any remainder.

Thus, if $a^{n-1} - b^{n-1}$ (the difference of the same powers of a and b) is exactly divisible by $a - b$, then the difference of the next higher power of $a^n - b^n$ is exactly divisible by $a - b$.

Hence if $a^2 - b^2$ is exactly divisible by $a - b$, then $a^3 - b^3$ is also exactly divisible by $a - b$, and so on indefinitely.

Illustrative example IV :

Prove by mathematical induction that

$$a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d].$$

Proof: By trial this formula for the sum of n terms is found to be true when $n = 2$, and when $n = 3$.

Let n have a value for which it is known to be true as $n = 3$, then for this value of n

$$a + (a + d) + \dots + [a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d] \text{ is true.}$$

Add $a + nd$ to both members of the equation, thus giving the series $(n + 1)$ terms.

$$\text{Then } a + (a + d) + \dots + [a + (n - 1)d] + (a + nd)$$

$$= \frac{n}{2}[2a + (n - 1)d] + (a + nd)$$

$$= an + \frac{dn^2}{2} - \frac{dn}{2} + a + nd$$

$$= \frac{2an + dn^2 + dn + 2a}{2}$$

$$= \frac{dn(n + 1) + 2a(n + 1)}{2}$$

$$= \frac{n + 1}{2}(2a + nd)$$

$$= \frac{n + 1}{2}(2a + [(n - 1) + 1]d).$$

This expression for $(n + 1)$ terms has the same *form* with reference to $(n + 1)$ that the original formula for the sum of n terms has with reference to n . Then if the formula is true for any number of terms, it is true for one more term. But it is true for $n = 3$, hence it is true for $n = 4$, therefore true for $n = 5$, and so on for all values of n .

Exercises

Prove by mathematical induction that

1. The sum of the first n even numbers is $n(n + 1)$.
2. The sum of the first n odd numbers equals n^2 .
3. $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$.
4. $2^2 + 4^2 + 6^2 + \dots + (2n)^2 = \frac{2n}{3}(n + 1)(2n + 1)$.
5. $1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n}{2}(n + 1) \right]^2$.
6. $1 + 3 + 3^2 + 3^3 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$.
7. $a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$.
8. $3 + 6 + 9 + \dots + 3n = \frac{3}{2}n(n + 1)$.
9. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1) = \frac{n}{3}(n + 1)(n + 2)$.
10.
$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{n(n + 1)(n + 2)}$$
$$= \frac{n(n + 3)}{4(n + 1)(n + 2)}$$
11. $x^{2n} - y^{2n}$ is exactly divisible by $x + y$ when n is a positive integer.
Hint: See illustrative example III and carry division until two terms are obtained in the quotient.
12. $x^n + y^n$ is exactly divisible by $x + y$ when n is odd.
13. $2 + 2^2 + 2^3 + 2^4 + \dots + 2^n = 2^{n+1} - 2$.
14. $3^{2n} - 1$ is exactly divisible by 8.
15. Illustrative example III by the remainder theorem.
16. The sum of $1^3 + 3^3 + 5^3 + \dots$ to n terms is $n^2(2n^2 - 1)$.

CHAPTER XVI

REVIEW

176. Miscellaneous Review. — A general review of the work is usually desired at the end of the course and the following exercises are offered to supplement the work of the preceding chapters.

Exercises

1. Represent graphically

(a) $2 + 3\sqrt{-1}$.

(b) The sum of $3 + 4\sqrt{-1}$ and $-2 - 3\sqrt{-1}$.

(c) The sum of $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, 1, and $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

2. Solve each of the following equations:

$$\frac{a^2}{x^2} + \frac{2a}{x} = 3.$$

$$ax - \frac{b}{x} = a - b.$$

3. Solve the following simultaneous equations:

$$x^2 + xy + 2y^2 = 11.$$

$$2x^2 - xy + 2y^2 = 8.$$

4. (a) How many different permutations can be formed from the letters x, x, x, y, z , and w , taken together?

(b) How many diagonals has a convex polygon of n sides? Check by drawing figures to illustrate the cases of $n = 3$ and $n = 4$.

5. By the use of graphs find to the nearest tenth each root of the following simultaneous equations:

$$y^2 - 2x = 4.$$

$$2x^2 + y = 3.$$

6. Find to three decimal places the root of the equation $x^3 + 5x^2 - 7x + 2 = 0$ that lies between 0.4 and 0.5.

7. Prove that an equation of the n th degree can have no more than n roots.

8. Solve the equation $x^3 + 3x^2 - 25x - 12 = 0$ for all roots. Check the roots.

9. All the roots of the equation $x^3 - 12x^2 + 47x - 60 = 0$ are real. Determine their signs.

10. Prove that the area of a square figure cannot be increased by increasing one dimension and decreasing the other by the same amount.

11. It is proved in geometry that $\overline{PT}^2 = PA \times PB$ where PT is the tangent from the external point P and where PB is any line from P meeting the circle in A and B ; if it is known that $PA = 2$ and $PB = 1 + PT$, what is the length of PT ?

12. If the edges of a rectangular box are increased by 2 inches, 3 inches, and 4 inches respectively, the box becomes a cube and the capacity is increased by 1008 cubic inches. Find the dimensions of the box before the increase.

13. (a) Transform the equation $x^3 - 6x^2 + 4x + 8 = 0$ into an equation of the form $x^3 + px + q = 0$.

(b) Solve the resulting equation and from these roots find the roots of the given equation.

(c) Check by forming the equation having this last set of roots.

14. By the use of logarithms find the value of

$$\frac{(.5246)^4 \times \sqrt[5]{-63427}}{\frac{1}{4}(.00386)^{\frac{1}{3}}}.$$

15. Find correct to the nearest tenth the positive root of the equation $x^3 + x^2 - 2x - 1 = 0$.

16. Prove that the equation $x^7 + x^4 - x^2 - 3 = 0$ has either 4 or 6 complex roots.

17. Solve the equation

$$3x^2 + 4x + 2\sqrt{x^2 + x + 3} = 30 - x^2.$$

18. A coach bought a certain number of footballs for \$121.50. If each ball had cost \$2 less, he could have bought one more ball for the money and still have had \$6.50. How many balls did he purchase?

19. One of the roots of the equation

$$8x^4 - 38x^3 + 71x^2 - 6x - 63 = 0 \text{ is } 2 + i\sqrt{3}.$$

Find the other roots.

20. (a) Five baseball nines arranged a schedule in which each team played with every other team twice. How many games did they schedule?

(b) In how many ways can 5 red books and 3 blue ones be arranged on a shelf so that all the books of each color will be together?

21. Find the equation whose roots are the roots of

$$3x^4 + 7x^3 - 15x^2 + x - 2 = 0$$

each increased by 2.

22. Solve by means of graphs the following set of equations:

$$y = x^2 - 5x + 6.$$

$$3x + 2y = 12.$$

23. Solve for x and y and correctly group your answers:

$$x^2 + y^2 - 13 = 0.$$

$$xy + y - x = -1.$$

24. (a) Rationalize the denominator of $\frac{\sqrt{3} + \sqrt{-4}}{2\sqrt{3} - 3\sqrt{-4}}$.

(b) Find graphically the sum of

$$4 - \sqrt{-3} \text{ and } 3 + 2\sqrt{-3}.$$

25. A merchant has 35 pounds of coffee that cost him 43 cents a pound. How many pounds of coffee costing 25 cents a pound must he mix with it to have a mixture that he could sell for 33 cents a pound and still gain 10%? (Compute as profit based on cost.)

26. Express both (a) and (b) in the form $c + di$ where $i = \sqrt{-1}$.

(a) $(3 - 2i)^2$.

(b) $\frac{4 - \sqrt{-9}}{1 - \sqrt{-4}}$.

Add the two resulting expressions and represent the sum graphically.

27. From the letters of the word "facetious" how many permutations of five letters each may be formed? How many words of five letters each may be formed from three of the consonants and two of the vowels if each arrangement of the letters is considered a word?

28. One root of the equation $x^4 - 12x^3 + 55x^2 - 114x + a = 0$ is $3 - \sqrt{-2}$. Find the other roots and the value of a .

29. Transform the equation $x^3 - 5x^2 - 2x + 24 = 0$ as follows:

(a) Multiply the roots by 2. (b) Diminish the roots by 3.

(c) Solve the resulting equation in (b) and from these results state the roots of the original equation.

30. Find the positive root of $x^3 - 5x^2 - 2x + 20 = 0$ to the nearest hundredth.

31. Given the equation $x^4 - 3x^2 - 5 = 0$; without solving, complete the following statements:

(a) The equation has () roots.

(b) It has () positive roots and () negative roots.

(c) It has () imaginary roots.

(d) The sum of the roots is ().

(e) The product of the roots is ().

32. (a) Plot the graph of $y = 3x^4 - 4x^3 - 12x^2 + 3$ from $x = -2$ to $x = 3$.

(b) From the graph read to the nearest tenth the roots of $3x^4 - 4x^3 - 12x^2 + 3 = 0$.

33. (a) Derive the fundamental formula for the sum of a finite geometric series in terms of a , r , and n .

(b) Insert three geometric means between 6 and 486.

34. The formula $D = \sqrt[3]{\frac{W}{.5236(A - G)}}$ gives the diameter

of a spheric balloon which is to lift a given weight W ; by the use of logarithms find D if $A = .0807$, $G = .0056$, $W = 1250$.

35. An open box is to be made from a rectangular piece of tin 20 inches long and 8 inches wide, by cutting out equal squares from the corners and turning up the sides. How large should these squares be in order that the box may contain 128 cubic inches?

36. In a race between two crews the first crew rows at a uniform rate of 360 yards per minute. The second crew starting at the same time covers 75 yards the first minute and for the next four minutes increases its rate 95 yards each minute; for the remainder of the race the crew maintains the rate that it has at the end of the fifth minute. In how many minutes will the second crew overtake the first?

37. Solve the equation $x^4 - 6x^3 + 19x^2 - 26x + 18 = 0$, knowing that one root is $2 - \sqrt{-5}$.

38. (a) Find the value of $\frac{2x - 3}{x^2 - x + 7}$ when $x = 2 - 3i$ and express the result in the form of $a + bi$.

(b) Add graphically $\frac{-1 + \sqrt{-3}}{2}$ and $\frac{-1 - \sqrt{-3}}{2}$.

39. (a) Derive the rule for transforming an equation into another whose roots are those of the first equation multiplied by the constant m .

(b) Form the equation with integral coefficients whose roots are the roots of $2x^4 + 10x^3 - 7x^2 + x + 4 = 0$ each multiplied by $\frac{1}{2}$.

40. (a) Without solving, determine the nature of the roots of the equation $x^5 + 4x^3 - 3x^2 - x + 12 = 0$.

(b) Write the value of the sum of the roots, the product of the roots, and the sum of the products taken three at a time.

41. Locate all the roots of $2x^3 + 3x^2 - 9x - 7 = 0$ and find the positive root to the nearest tenth.

42. (a) How many different committees consisting of 3 seniors and 2 juniors can be selected from 8 seniors and 10 juniors?

(b) If ${}_nP_r = 20$ and ${}_nC_r = 10$, find n and r .

43. Find the sum of all integers between 100 and 800 that are divisible by 3.

44. Find what values of k will make the two values of x equal in the solution of the simultaneous equations:

$$\begin{aligned} 16x^2 + 25y^2 &= 400, \\ y &= \frac{4}{5}x + k. \end{aligned}$$

45. Two launches race over a course of 12 miles. The first travels at the rate of $7\frac{1}{2}$ miles per hour. The other, which has a start of 10 minutes, runs over the first half of the course with a certain speed but increases its speed over the

second half by 2 miles per hour, winning the race by a minute. Find the speed of the second launch over the first half of the course.

46. (a) Plot the graph of $y = x^3 - 9x^2 + 24x - 7$ from $x = 0$ to $x = 6$.

(b) From the graph determine a root of

$$x^3 - 9x^2 + 24x - 7 = 0.$$

(c) From the graph determine the roots of

$$x^3 - 9x^2 + 24x - 7 = 13.$$

(d) From the graph determine the roots of

$$x^3 - 9x^2 + 24x - 7 = 25.$$

47. (a) Write the first four terms in the expansion of

$$\left(\frac{x^2}{2} - 2y\right)^{10}.$$

(b) Without expanding, find the eighth term and simplify the result.

48. Given the polynomial $2x^3 - 6x^2 + 100$; calculate its value when $x = 4 + 3i$ and represent the result graphically. ($i = \sqrt{-1}$.)

49. Solve the following equation for the value of x :

$$577 = 455(1.02)^{2x}.$$

50. (a) Derive a formula for the sum of n terms of an infinite decreasing geometric progression in terms of a and r .

(b) Express as an ordinary fraction the repeating decimal $0.1313 \dots$

51. (a) State Descartes' rule of signs for positive and negative roots of an equation.

(b) By the use of this rule and by inspection of the constant term, obtain all the information possible concerning the roots of $4x^5 - 2x - 1 = 0$.

(c) Without expanding $(2x^2 - \frac{1}{2}x^{-1})^{10}$, find the seventh term and simplify the result.

52. (a) Transform $4x^3 - 12x^2 - x + 3 = 0$ into an equation with integral coefficients, the coefficient of the highest degree term being unity.

(b) Solve the resulting equation and from these results write the roots of the original equation.

53. One root of the equation $x^3 - 9x^2 + kx - 65 = 0$ is $2 - 3i$. Find the other roots and the value of k .

54. Find the positive root of $x^3 + 9x^2 + 27x - 50 = 0$ correct to the nearest hundredth.

55. (a) Plot the graph of $y = \frac{1}{2}x^3 - x^2 - 2x + 2$ from $x = -2$ to $x = +3$.

(b) From the graph estimate the roots of

$$\frac{1}{2}x^3 - x^2 - 2x + 2 = 0.$$

(c) From the graph determine the roots of

$$\frac{1}{2}x^3 - x^2 - 2x + 2 = -2.$$

56. A box contains 7 red cards, 6 white cards, and 4 blue cards. How many selections of three cards can be made so that (a) all three are red, (b) none are red?

57. A certain principal and interest for one year amount to \$132.50. If the principal were \$25 more and the rate of interest 1% less, the amount in one year would be \$157.50. Find the principal and the rate of interest.

58. If one root of $x^4 - 2x^3 + x^2 + 6x + 14 = 0$ is $2 + \sqrt{-3}$, find the other roots.

59. Expand by the Binomial Theorem and simplify, expressing the result with positive exponents and rational denominators:

$$\left(\sqrt{2a} + \frac{1}{2\sqrt[3]{-2a}}\right)^4.$$

60. (a) Prove that if a set of numbers are in geometric progression, their logarithms are in arithmetic progression.

(b) By the use of a formula determine the number of terms that must be taken in the series 2, 5, 8, 11 $\cdots$ so that the sum shall be 345.

61. (a) Plot the graph of $y = 3x^3 - 16x^2 + 15x + 18$ from $x = -1$ to $x = +4$.

(b) From the graph determine the roots of

$$3x^3 - 16x^2 + 15x + 18 = 0.$$

(c) From the graph determine approximately the roots of

$$3x^3 - 16x^2 + 15x = -8.$$

(d) From the graph determine the nature of the roots of

$$3x^3 - 16x^2 + 15x = -30.$$

62. Solve for x and y and group your answers:

$$x^2 + 3xy = 18.$$

$$x^2 - 5y^2 = 4.$$

63. (a) Obtain all the information possible concerning the roots of the following equation by the use of Descartes' rules and by inspection of the constant term:

$$x^5 - 3x + 2 = 0.$$

(b) For what values of k will the roots of

$$x^2 + kx + (k - 3) = 0$$
 be equal?

64. (a) How many arrangements of the letters of the word *logarithm* can be made, if all the letters are used at a time and if each arrangement begins with a vowel and ends with a consonant?

(b) How many arrangements, each consisting of three consonants and two vowels, can be formed from the same word?

65. The formula for the amount (A) of P dollars at 6% interest for n years, compounded quarterly, is $A = P(1.015)^{4n}$.

Find the length of time it will take \$524 to amount to \$1146, if the rate of interest is 6% and if the interest is compounded quarterly.

66. By the use of Horner's method find to the nearest tenth the positive root of $x^3 - x^2 + x - 10 = 0$.

67. Two towns, A and B , are 60 miles apart. An automobile starts from A and proceeds to B at the rate of 20 miles an hour. A second automobile starts from A 20 minutes after the first and goes the first half of the distance at a certain rate and the second half of the distance at a rate 10 miles per hour faster, arriving at B 10 minutes ahead of the first automobile. Find the original rate of the second automobile.

68. Solve graphically the following set of equations:

$$\begin{aligned}x^2 + 4y^2 &= 100, \\y &= x^2 - 2x - 3.\end{aligned}$$

69. The first term of an arithmetic progression is 2, and the first, third, and eleventh terms are also the first three terms of a geometric progression. Find the sum of the first eleven terms of the arithmetic progression.

70. Verify by substitution that $1 + \sqrt{-2}$ is a root of the equation $2x^3 - 7x^2 + 12x - 9 = 0$. Find the other roots. Add graphically the three roots of the equation.

71. Transform the equation $2x^3 - 3x^2 - 4x + 6 = 0$ into an equation in which the coefficients are integers, that of the highest degree term being unity. Find the roots of the resulting equation and of the given equation.

72. Find two values of y for which the equation

$$x^3 - 6x^2 + 9x - y = 0 \text{ has two equal roots.}$$

73. (a) In how many years will a sum of money double itself at 6% if the interest is compounded annually?

(b) Given $p = \left(\frac{2}{x+1} \right)^{\frac{x}{x+1}}$; find the value of p if $x = 1.41$.

74. (a) How many different sums of money can be made from a cent, a nickel, a dime, a quarter, a half dollar, and a dollar?

(b) How many different signals can be made with six flags of different colors if three are used at a time in a vertical line?

75. By the use of Horner's method, find to the nearest tenth the positive root of

$$x^3 + 2x^2 - 5x - 9 = 0.$$

76. Two motor boats race 25 miles down a river and the same distance back; at the end of the first half hour one boat is three quarters of a mile ahead, and later it meets the other boat one mile from the turning point. Find the rates of the boats in still water, if the rate of the river is 3 miles an hour and if both boats can make headway upstream.

77. Find the value of k that will make the two values of x equal in the solution of the simultaneous equations

$$y = 2x + k.$$

$$y = x^2 - 5x + 6.$$

(This is the condition that will make the straight line become tangent to the parabola.)

78. (a) Plot between the values $x = -3$ and $x = +5$ the curve represented by the following equation

$$y = x^3 - 3x^2 - 9x + 15.$$

(b) From the curve estimate to the nearest tenth the x intercepts.

79. Assuming that every rational equation has a root, prove that an equation of the n th degree has exactly n roots.

80. Three numbers whose sum is 24 are in arithmetic progression; if 2, 6, 17 are added to them respectively, the results are in geometric progression. Find the numbers.

81. Form the equation of the fourth degree with rational coefficients, three of whose roots are 2, -3 , $3 + 2\sqrt{-1}$.

82. In a certain county there are 15 candidates for State scholarships. In how many ways may 5 scholarships be awarded to 2 girls and 3 boys if 6 of the candidates are girls and 9 are boys?

83. The distance through which a body falls varies as the square of the time in falling. A stone dropped from a window 36 feet 1 inch above the ground strikes the ground in $1\frac{1}{2}$ seconds; one dropped from a bridge strikes the water below in $4\frac{1}{2}$ seconds. Find the height of the bridge above the water.

84. Plot the graph of $2x^3 + 3x^2 - 10x - 7 = y$ and from the graph determine the location of the roots of the equation formed by making $y = 0$.

85. Given $\log 2 = 0.30103$, $\log 3 = 0.47712$, $\log 7 = 0.84510$; find the logarithms of 84, 81, $\sqrt{7}$, 500, $\frac{3}{4}$.

86. By Horner's method of approximation find the root, correct to two decimal places, between 1 and 2 of

$$x^3 - 9x^2 + 23x - 16 = 0.$$

87. By determinants find the value of x in the following system of equations:

$$x + 2y + 3z = 1.$$

$$2x - y - 2z = 6.$$

$$3x + 3y - z = -5.$$

88. Extract the square root of $-12 + 16\sqrt{-1}$.

89. Represent graphically and construct the sum of the following complex numbers:

$$\frac{1 + 18i}{3 + 4i} \text{ and } \frac{3 - 29i}{3 - 4i}. \quad (i = \sqrt{-1})$$

90. Factor the determinant $\begin{vmatrix} c & x & x \\ d & x & a \\ c & b & b \end{vmatrix}$.

91. By determinants solve the following equations for y only:

$$5x - 2y + 2z = 2.$$

$$4x + 5y - 3z = 7.$$

$$5x - 2y - 3z = -4.$$

92. How many different signals can be made with 10 flags, 2 red, 3 blue, and the rest white, if all are used for each signal and are hoisted one above another?

93. Determine the number of positive, negative, and imaginary roots of the following equation:

$$x^4 + 15x^2 + 7x - 40 = 0.$$

94. Locate graphically the roots of the equation

$$x^4 - 2x^3 - 9x^2 + 10x + 5 = 0.$$

Determine approximately from the graph the minimum value or values of the first member of the equation.

95. One root of the equation $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$ is $2 - \sqrt{3}$. Find the other roots.

96. Find by Horner's method, to two decimal places, the positive root of

$$x^3 + 3x^2 + 5x - 178 = 0.$$

97. By the binomial formula find the cube root of 30 to three places of decimals.

98. Resolve into partial fractions

$$\frac{2x^2}{(x+1)^2(x^2+1)}.$$

99. Given $\log 2 = 0.30103$; solve the exponential equation

$$4^{x-1} = \sqrt{10}.$$

100. Find the number of inscribed triangles that can be formed by joining 15 points on the circumference of a circle.

101. Derive the formula for finding P permutations of n things taken r at a time.

102. Find the roots of $4x^2 + 8x + 5 = 0$ and represent them graphically.

103. Prove that the interchange of any two adjacent columns of a determinant changes the sign of the determinant.

104. Solve by determinants

$$x + 2y + 3z = 13.$$

$$2x + y + z = 7.$$

$$3x + 4y + 3z = 21.$$

105. Show that the equation $x^3 + px + q = 0$ has but one real root if p and q are positive, and that such real root is negative.

106. In the equation $x^3 - 3x - 4 = 0$, find by Horner's method, to two decimal places, the root lying between 2 and 3.

107. Construct from $x = -2$ to $x = +3$ the graph of the equation $x^3 - 2x^2 - 2x + 3 = y$ and from this graph locate the roots of the equation.

108. The roots of the equation $4x^3 - 12x^2 + 11x - 3 = 0$ are in arithmetical progression. Find these roots.

109. Transform the equation $5x^3 - 6x^2 - 3x + 8 = 0$ into an equation with integral coefficients, the coefficient

of the highest power being unity. How are the roots of the transformed equation related to those of the given equation?

110. How many different sums of money can be made up from a cent, a nickel, a dime, and a quarter? (Solve by permutations and combination.)

111. What is the distinction between a permutation and a combination? Deduce the formula for the number of combinations of n things taken r at a time.

112. Show that $(2 + i)^2 = \frac{7 + i}{1 - i}$ when $i = \sqrt{-1}$.

113. How many terms are there in the development of a determinant of the n th order? How are these terms formed? How is the sign of any term determined?

114. Find by the use of determinants the value of y in the following system of equations:

$$x + 3y - z = 1.$$

$$y + 3z - w = 4.$$

$$z + 3w - x = 11.$$

$$w + 3x - y = 2.$$

115. Without solving the equation, state five facts in regard to the roots of the equation $x^5 + 2x^3 + x^2 - x - 1 = 0$.

116. Find all the roots of the equation

$$x^4 - 6x^3 + 6x^2 + 10x - 3 = 0.$$

117. Plot for integral values of x between $x = 0$ and $x = 5$ the equation $y = x^3 - 2x^2 - 4x - 5$ and determine, to two places of decimals, the positive root of the right-hand member set equal to zero.

118. An equation with real and rational coefficients is known to have as roots the numbers $\sqrt{2}$, 3 , $1 - i$. What other roots must this equation have? Assuming that this

equation is of the lowest possible degree, find the coefficient of the x term in next to the highest power of x and find the constant term.

119. If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio of $m : n$, show that $mnb^2 = (m + n)^2ac$.

120. For what values of k are the roots of the equation

$$x^2 + k(x + 1) + 3 = 0$$

(a) complex, (b) real, (c) equal?

121. Two bodies, A and B, are moving at a constant rate and in the same direction around the circumference of a circle whose length is 20; A makes one circuit in two seconds less time than B and at the end of one minute A has made one circuit more than B. At what rates are A and B moving?

122. Deduce the formula for the number of permutations of n things taken all at a time, when p are of one kind, q of another kind, and r of a third kind.

123. Prove the relation $(1 - i)^3 = \frac{-4i}{1 + i}$. ($i = \sqrt{-1}$.)

124. Solve the following for y , using determinants of the 4th order:

$$x + y + z + w = 0.$$

$$x + 2y + 3z + 4w = 0.$$

$$x + 3y + 6z + 10w = 0.$$

$$x + 4y + 10z + 20w = -1.$$

125. Solve $x^4 - 22x^2 + 12x + 48 = 0$.

126. Write, with integral coefficients, the equation of lowest degree two of whose roots are $\sqrt{2}$ and $-6\sqrt{-1}$.

127. Assuming that every rational integral equation in one variable has one root, prove that it has no more than n roots.

128. State Descartes' rule of signs. By means of this rule show that the equation $x^4 + ax + b = 0$ cannot have four real roots unless a and b both equal zero.

129. Plot the equation $x^3 + 7x^2 - 11x - 2 = y$. Compute by Horner's method to two places of decimals the positive root of the left member of this equation set equal to zero.

130. (a) Prove that the sum of the roots of the quadratic equation $x^2 + px + q = 0$ is equal to $-p$ and that the product of the roots is equal to q .

(b) In the equation $ax^2 - 3x + k = 0$, what must be the value of k in order that the product of the roots shall be twice that of their sum?

131. (a) State the condition under which a quadratic equation will have one root equal to zero.

(b) Determine k so that the equation $(x - k)^2 + 3(x - 2k) = 0$ shall have one root equal to zero.

132. Find all the roots of the equation $x^3 - 1 = 0$ and represent them graphically. Show graphically that the sum of the roots of the equation is equal to zero.

133. Find the number of distinct permutations of the letters of the word "committee," using all the letters in each permutation.

134. Evaluate the determinant
$$\begin{vmatrix} 0 & -1 & -1 & 1 \\ 4 & 5 & 1 & 1 \\ 3 & 9 & 4 & 1 \\ -4 & 4 & 4 & 1 \end{vmatrix}.$$

135. Solve the equation $x^5 - 12x^3 + 46x^2 - 85x + 50 = 0$, two of whose roots are $1, 1 + 2i$.

136. The edges of four cubes are in arithmetical progression, having a common difference of one inch. If the largest

cube is equal in volume to the sum of the other three, find their edges.

137. Deduce the rule for finding an equation whose roots are equal to the roots of a given equation, each multiplied by a constant k . What use do you make of this method in advanced algebra?

138. Solve the equation

$$4x^3 - 16x^2 + 19x - 6 = 0.$$

139. Plot the equation $x^3 + 3x^2 - 2x - 1 = y$. Compute by Horner's method, to two places of decimals, the positive root of the left-hand member of this equation set equal to zero.

140. Given $(9x + 5k^2)(x + k) = 2x$.

Determine k so that (a) the roots may be equal numerically but opposite in sign, (b) one root may equal 0.

141. (a) What value must k have in order that one root of the equation $x^2 - 3kx - 2k + 4 = 0$ may equal 2?

(b) Find the equation whose roots are double the roots of

$$x^2 + ax + b = 0.$$

142. Show by substitution that $1 + i$ is a root of the following:

$$2x^3 - x^2 - 2x + 6 = 0.$$

143. Express the following sum as a single determinant and evaluate it:

$$\begin{vmatrix} 15 & 2 & 6 & 1 \\ -5 & 0 & 4 & 8 \\ 2 & 7 & 3 & 10 \\ 8 & 8 & 2 & 2 \end{vmatrix} + \begin{vmatrix} -10 & 2 & 6 & 1 \\ 5 & 0 & 4 & 8 \\ 7 & 7 & 3 & 10 \\ 2 & 8 & 2 & 2 \end{vmatrix}.$$

144. How many parallelograms are formed when a set of 10 parallel lines intersects another set of 8 parallel lines?

145. Form an equation of the fourth degree, two of whose roots are $1 + 2i$ and $-\sqrt{2}$.

146. Solve $x^3 - 12x^2 + 23x + 36 = 0$ if the roots are in arithmetical progression.

147. Prove that if a is a root of the equation $f(x) = 0$, then $x - a$ is a factor of $f(x)$.

148. Solve $3x^3 + 16x^2 + 18x - 20 = 0$.

149. Plot the equation $x^3 - 2x - 2 = y$. Compute by Horner's method, to two places of decimals, the positive root of the left member of this equation set equal to zero. What is the nature of the remaining roots?

150. For what values of k does the equation $(x + k)x - (k - 3) = 0$ have its roots equal? Give a value of k that makes both roots rational.

151. Determine k and m so that the equation

$$x^2 + 2kx + 3mx - k + 2m + 7 = 0$$

shall have both roots equal to zero.

152. What is the value of $x^3 + 3x^2 - 7x + 10$ when $x = 1 - i$? ($i = \sqrt{-1}$.)

153. In the system of equations

$$\begin{aligned}2x + 3y - z &= 0 \\2x - 3y + 3z &= 2 \\-x + 2y + 5z &= 5\end{aligned}$$

find the value of z by the use of determinants.

154. (a) In how many ways can 7 boys stand in line, only 2 being willing to stand at the extremities of the line?

(b) Of 8 books of the same size, a shelf will hold 5. How many different arrangements may be made on the shelf?

155. (a) Prove that if all of the elements of a row of a determinant are zero, the value of the determinant is zero.

(b) Solve $x(x+1)(x+2)(x+3) = 24$.

156. (a) What information regarding the roots of the equation $x^5 - 4x^4 + 2x + 1 = 0$ is obtainable by the application of Descartes' rule?

(b) From a sketch of the graph of the equation

$$f(x) = x^5 - 4x^4 + 2x + 1 = y$$

estimate the values of the real roots of $f(x) = 0$.

157. It is desired to double the capacity of a tank $3 \times 4 \times 5$ feet by making equal elongations of its dimensions. Find the elongation of each dimension.

158. Prove that if a rational integral equation with real coefficients has the complex number $c + id$ for a root, it must also have the number $c - id$ for a root.

159. Compute by Horner's method, to two decimal places, one root of the equation

$$x^4 + 2x^2 - 2x - 4 = 0.$$

160. (a) Plot the graph of $2x^3 + 8x^2 - 8$.

(b) If it were required to find from the graph the real roots of the equation $2x^3 + 8x^2 - 8 = 0$, explain why it would be unnecessary to extend the table of values beyond $x = -4$ and $x = 1$.

161. Find to two decimal places the real root of the equation

$$x^3 + 2x - 4 = 0.$$

162. Given the equation

$$x^4 - 2x^3 + 4x - 4 = 0.$$

Without solving the equation, fill out the following statements by placing the proper number in each parenthesis;

justify each statement by quoting the appropriate theorem or principle :

- (a) The equation has () roots.
- (b) It has () positive roots and () negative roots.
- (c) It has () fractional roots.
- (d) It has at least () imaginary roots.
- (e) All the integral roots, if any, must be factors of ().
- (f) The product of the roots is ().

163. How long is the edge of a cube, if the number of square feet in its surface exceeds the number of cubic feet in its volume by 27? (Solve either algebraically or graphically.)

164. Given the equation $ax^2 + bx + c = 0$. What can you tell about the character of the roots if a and c have opposite signs? Does the sign of b affect the character of the roots? What must be true if one root is to be zero?

165. (a) Prove the theorem: If a is a root of $f(x) = 0$, then $x - a$ is a factor of $f(x)$.

(b) Prove that $x - 1$ is a factor of $100x^7 - 101x^3 + 1$.

166. (a) Simplify $\frac{1 + 2i + 3i^2}{1 - 2i + 3i^2}$.

Represent the result graphically.

(b) Find the three cube roots of 8 and show graphically that their sum is 0.

167. Given the formula

$$\frac{wv^2e}{2g} = \frac{tr}{12}.$$

Express each of the variables w , v , and t , in terms of the other variables.

168. A freight boat leaves New York, sailing for a European port at the rate of 15 knots an hour. Five hours later a transport leaves the same harbor, sailing over the same

course at the rate of 20 knots an hour. The freight boat, two hours after its departure, is delayed one hour by a defect in its machinery. In how many hours will the transport overtake the freight boat? (Solve either algebraically or graphically.)

169. From 16 soldiers in how many ways can a guard of 5 be chosen? In how many ways can a guard of 7 chosen from the 16 soldiers be arranged in line? In how many ways can the 16 be divided into two equal groups?

170. A signal corps has six different flags. By using one, or two, or three flags at a time, how many different signals can be formed with these flags?

171. A rectangular sheet of cardboard, 9" by 12", is to be made into an open box, by cutting out a square from each corner and turning up the sides. If the volume of the box is to be 80 cubic inches, find the length of a side of each square cut out.

172. If d is the distance in yards at which a certain train can be stopped on the level when going at a speed of V miles an hour, it is known that the following relation exists:

$V =$	30	40	55	60
$d =$	100	180	340	400

Plot a curve expressing this relation and from it find d when $V = 35; 45; 50$.

173. Given the quadratic equation

$$S = 111.7 D + 1.35 D^2 - 110.97.$$

Find to two significant figures the value of D when $S = 59.62$.

174. A bomb dropped from a point H feet above the earth, by an airplane moving s feet per second, will fall D feet ahead

of the perpendicular on which it was dropped, D being found by the formula

$$D = \left(\frac{\sqrt{H}}{4} + \frac{H}{800} \right) s - \frac{H}{40}.$$

If it is known that $s = 100$ and $D = 2000$ feet, find the height of the airplane to the nearest 100 feet.

175. Find all the roots of the equation $x^6 - 1 = 0$. Check each root.

176. (a) Plot the graph of $x^3 - 15x - 4$.

(b) From this graph find to two significant figures the value of x in the equation $x^3 - 15x - 4 = 0$ that lies between -3 and -4 .

177. Solve the equation $x^4 + 2x^3 + 2x^2 - 2x - 3 = 0$.

178. Find to two decimal places the positive root of the equation

$$x^3 - 2x - 5 = 0.$$

179. Given the equation

$$x^n + c_1x^{n-1} + c_2x^{n-2} + \dots + c_n = 0.$$

(a) What is the sum of the roots?

(b) What is the product of the roots?

(c) Express c_2 in terms of the roots.

180. One of the roots of the equation

$$x^4 - 2x^2 + 8x - 3 = 0 \text{ is } 1 - \sqrt{-2}.$$

Find the other roots.

181. In how many ways can a detail of 2 officers and 5 men be chosen from a group of 10 officers and 100 men?

182. (a) If the equation $x^2 + 2(1+k)x + k^2 = 0$ has equal roots, find the value of k .

(b) Show that the equation $3mx^2 - (2m + 3n)x + 2n = 0$ has rational roots.

183. Given the equation

$$4x^5 + 3x^3 - 40x^2 - x + 10 = 0.$$

Without solving the equation, fill out the following statements by placing the proper number in each parenthesis; justify each statement by quoting the appropriate theorem or principle:

(a) The equation has () roots.

(b) It has no more than () positive roots and no more than () negative roots.

(c) It has at least () fractional roots.

(d) It has at least () imaginary roots.

(e) All the integral roots, if any, must be factors of ().

(f) The product of the roots is ().

(g) The equation has at least () real roots.

184. Solve and check

$$\frac{x}{x-5} - \frac{x-5}{x} = \frac{3}{2}.$$

185. For what values of m will the equation

$$\frac{x^2 - x + m - 1}{(x-1)(m-1)} = \frac{x}{m}$$

have two equal values for x ?

186. (a) Rationalize the denominator of

$$\frac{x - \sqrt{x^2 - 1}}{x + \sqrt{x^2 - 1}}.$$

(b) Find the value of this expression when $x = 4$, evaluating the radical to the nearest hundredth.

187. Find by formula the sum of the first 6 terms of the progression

$$2\frac{2}{5}, 3\frac{1}{5}, 4\frac{4}{15} \dots$$

188. Using logarithms, find the value of n in the formula

$$n = \frac{1}{2L} \sqrt{\frac{Mg}{m}} \text{ when } L=78.5, M=5468, g=980, m=0.0065.$$

189. Find all the roots of $x^4 + x^3 - 2x^2 + 4x - 24 = 0$. Check by forming the product of the roots and comparing it with the proper term in the equation.

190. Given the equation

$$3x^3 - x^2 - 5 = 0.$$

(a) Write the sum of the roots and the product of the roots.

(b) Transform to an equation whose roots are twice the roots of the given equation.

(c) Transform to an equation with integral coefficients, the coefficient of the highest degree term being unity.

191. (a) Plot the graph of $x^3 - 7x + 6$ from $x = -4$ to $x = +4$.

(b) What is the greatest value of this expression for values of x between -4 and $+3$?

(c) From the graph determine the roots of the equation

$$x^3 - 7x + 6 = 0.$$

192. Find to the nearest hundredth the root of the equation $x^3 + 3x^2 - 4x - 1 = 0$, which lies between 1 and 2.

193. (a) Draw the graph of each of the following and of their sum:

$$3 + \sqrt{-2}, 3 - \sqrt{-2}.$$

(b) In the expression $1 + ix + \frac{i^2 x^2}{2} + \frac{i^3 x^3}{3!}$

$x = 2$ and $i = \sqrt{-1}$. Find the value of this expression. ($n!$ = factorial n .)

194. In a league of 10 basketball teams each team plays two games with every other team. How many weeks will

the playing season last if three games are played each week? How many times will any one team play?

195. A formula for the flow of water in a long horizontal pipe connected with the bottom of a reservoir is

$$\frac{Hd}{L} = \frac{4v^2 + 5v - 2}{1200}$$

when H is the depth of the water in the reservoir in feet, d the diameter of the pipe in inches, L the length of the pipe in feet, and v the velocity of the water in feet per second. If a reservoir contains 49 feet of water, find the velocity of the water in a 5-inch pipe that is 1000 feet long.

196. The so-called effective area of a chimney is given by the formula $E = A - \frac{3}{5}\sqrt{A}$ when A is the measured area. Solve this equation for A in terms of E .

197. (a) Express $(1 + i)^3$ and $\frac{2 + 2i}{1 - i}$ in the form $a + bi$.

(b) Represent graphically $2i - 2$ and $1 + 3i$ and find their sum graphically.

198. If one of the roots of $6x^4 - 11x^3 + x^2 + 33x - 45 = 0$ is $1 - \sqrt{-2}$, find the other roots.

199. Find the values of q which will make two roots of the equation $x^3 - 3x + q = 0$ equal.

200. By the use of Horner's method find to the nearest hundredth the positive root of $x^3 + x - 4 = 0$.

201. (a) If no two books are alike, in how many ways can 2 red books, 3 green books, and 4 brown books be arranged on a shelf so that the books of the same color are together?

(b) Six baseball nines wish to arrange a schedule of games in which each nine shall meet every other nine three times. How many games must be scheduled?

202. (a) Express as a common fraction the repeating decimal $0.42727\ldots$.

(b) Solve the following equation for n :

$$876 = 329(1.06)^{4n}$$

203. Solve the following set of equations and group your answers:

$$\frac{1}{x^3} + \frac{1}{y^3} = 35.$$

$$\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 7.$$

204. (a) Obtain all the information possible concerning the roots of the following equation by the use of Descartes' rules and by inspection of the constant term:

$$x^4 + x^3 - x^2 + x - 1 = 0.$$

(b) Transform $x^3 - 2x^2 + 5x + 7 = 0$ into an equation whose roots are -2 times the roots of the given equation.

205. A and B can run around a circular track 880 yards in length in 3 minutes and 7 minutes respectively. If they start together, in how many minutes will A overtake B?

206. (a) Plot between the values $x = -2$ and $x = +3$ the curve represented by the equation $y = 4x^3 - 6x^2 - 6x + 10$.

(b) From the curve estimate to the nearest tenth the negative root of the equation $4x^3 - 6x^2 - 6x + 10 = 0$ and the positive value of x that will produce the least value of $4x^3 - 6x^2 - 6x + 10$.

207. The sum of the first two terms of a decreasing geometric series is $\frac{5}{4}$; the sum to infinity is $\frac{9}{4}$. Find the first three terms of the series.

208. Find the roots of the equation

$$3x^3 - 4x^2 + 2x + 4 = 0.$$

209. (a) Solve the equation $x^3 - 1 = 0$ and check one of the complex roots by substitution.

(b) Represent the roots graphically and find their sum graphically.

(c) Check the result of the graphic addition by using one of the relations between the roots and the coefficients.

210. (a) Transform the equation $x^3 - 6x^2 + 6x + 4 = 0$ into another equation of the form $x^3 + px + q = 0$.

(b) Solve the resulting equation and from these roots obtain the roots of the given equation.

(c) Check by forming the equation having this last set of roots.

211. (a) Find the value of $\left(\frac{0.004683}{0.06787}\right)^{\frac{4}{5}}$.

(b) If interest is compounded m times a year at the annual rate r , the amount A of P dollars after n years is given by the formula $A = P\left(1 + \frac{r}{m}\right)^{mn}$. Find the length of time it will take a sum of money to double itself if $r = 6\%$ and $m = 2$.

212. A cork sphere 2 inches in diameter floating in water sinks to a depth x given by the equation $2x^3 - 6x^2 + 1.92 = 0$. Find the depth correct to the *nearest hundredth*.

213. In how many ways can 8 men be arranged in a line if two of them, A and B, always stand together?

214. How many baseball nines can be chosen from 13 candidates, provided A, B, C, and D are the only candidates for two positions and can play in no other positions?

215. Find the values of k that will make two values of x equal in the solution of the simultaneous equations $y = x^3$ and $y = 3x + k$.

216. (a) Prove that the equation $x^5 + 2x^4 - x^2 - 10 = 0$ has either 2 or 4 complex roots.

(b) Find the 7th term of the expansion $\left(a - \frac{1}{\sqrt{a}}\right)^{10}$.

217. Solve the equations:

$$\begin{aligned}x^3 - y^3 &= 19. \\x^2y - xy^2 &= 6.\end{aligned}$$

218. The circumferences of the wheels of a carriage are in the ratio 4 : 5. In going 1 mile the fore wheel makes 80 revolutions more than the hind wheel. Find the circumference of each wheel.

Arranged Review

1. Simplify

$$\frac{\frac{a+b}{ab}\left(\frac{1}{a} - \frac{1}{b}\right) - \frac{b+c}{bc}\left(\frac{1}{c} - \frac{1}{b}\right)}{\frac{a+c}{ac}\left(\frac{1}{a} - \frac{1}{c}\right)}.$$

2. Solve

$$(a) \quad \frac{x - \frac{1}{b}}{d} + \frac{x - \frac{1}{c}}{b} + \frac{x - \frac{1}{d}}{c} = 0.$$

$$(b) \quad \frac{x-a}{b} + \frac{x+b}{a} = \frac{a+b}{a}.$$

3. Simplify

$$\frac{9y^2 - (4z - 2x)^2}{(2x + 3y)^2 - 16z^2} + \frac{16z^2 - (2x - 3y)^2}{(3y + 4z)^2 - 4x^2} + \frac{4x^2 - (3y - 4z)^2}{(4z + 2x)^2 - 9y^2}.$$

4. In the fraction $\frac{a}{b}$, b is always to equal H while a is to change its value.

(a) If a changes from $+1$ to 0 , does the value of $\frac{a}{b}$ increase or decrease? Between what two values?

(b) If a changes from 0 to -1 , does the value of $\frac{a}{b}$ increase or decrease? Between what two values?

(c) Answer similar questions when a changes from $z-1$ to 0 ; when it changes from 0 to $+1$.

5. If $a = 3 + \sqrt{3}$, $b = 2\sqrt{3}$, $c = \sqrt{6}$, find the value of $\frac{a^2 + c^2 - b^2}{2ac}$ in its simplest form.

6. John agrees to go, if Henry goes. John went. What did Henry do?

7. Factor $x^8 - 2x^4y^4 + y^8 - 4x^2y^2(x^2 - y^2)^2$.

8. A man goes to a store, and says: "Give me as many cents as I have, and I will spend a cents with you." It is done, and the man repeats the operation at a second and a third store, when he has no money remaining. What is the least number of cents he could have spent each time to have every transaction made in an integral number of cents?

9. The sum of two numbers is a , and the difference of the squares is k^2 . What are the numbers? Interpret your result (1) if $k^2 > a^2$; (2) if $k^2 = a^2$; (3) if $k^2 < a^2$.

10. State the condition that will make the product less than the multiplicand.

11. If one of the factors of 7 is $\sqrt{9+x} + \sqrt{2+x}$, what is the other factor?

12. Solve the system

$$\begin{aligned}x + xy + y &= 29. \\x^2 + xy + y^2 &= 61.\end{aligned}$$

13. From a thread whose length is equal to the perimeter of a square 36 inches are cut off, and the remainder is equal in length to the perimeter of another square whose area is four ninths of that of the first. What is the length of the thread?

14. Find the value of

$$\frac{x + .28}{.012} + \frac{\frac{x}{100} + .0028}{12} - \frac{\frac{x}{10} + \frac{28}{100}}{1200} + \frac{100x + 28}{1.2}$$

when $x = 17$, correct to four places of decimals.

15. Factor $a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2)$.

16. Two fixed points X and Y are located on a straight line 100 feet apart. A movable point, P , moves from X toward Y in the following uniform manner: during the first half of each minute P moves towards Y 3 feet, and during the second half of each minute it returns toward X 2 feet. In how many minutes will P reach Y ?

17. Solve for x , y , and z .

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9.$$

$$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 3.$$

$$\frac{1}{x} + \frac{1}{y} - \frac{1}{z} = 1.$$

18. In a half-mile race A can beat B by 20 yards and C by 30 yards. By how many yards can B beat C ?

19. Factor

(a) $x^3 - y^3 - 2x^2y + 2xy^2$.

(b) $x^2 + 9 - 2x(3 + 2xy^2)$.

(c) $a^2(a^2 - 1) - b^2(b^2 - 1)$.

(d) $ab(x^2 - y^2) + xy(a^2 - b^2)$.

20. A number of 2 digits is equal to seven times the sum of its digits. Show that one digit must be twice the other.

21. A certain kind of wine contains 20% alcohol and another kind contains 28%. How many gallons of each must be used to form 50 gallons of a mixture containing 21.6% alcohol?

22. For what value of c will $2x^2 - x + c$ and $4x^2 + 4x + 1$ have a common factor?

23. Solve the equation

$$\sqrt{2x+8} - 2\sqrt{x+5} = 2.$$

What sign must each radical have in order that your results may check the equation?

24. Simplify

$$\frac{a + \sqrt{b}}{\left[\sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}} \right]^2}$$

25. Solve and check:

$$\sqrt{\frac{3x+6}{7x-3}} + \sqrt{\frac{7x-3}{3x+6}} = \frac{13}{6}.$$

26. Simplify $\sqrt{(e^x + e^{-x})^2 - 4}$.

27. Simplify $\frac{x + \sqrt{x^2 + a^2}}{x - \sqrt{x^2 + a^2}} + \frac{x^2 + a^2 - x\sqrt{x^2 + a^2}}{a^2}$.

28. Find the value of a^9b^9 if $a = x^{\frac{5}{9}}y^{-\frac{5}{9}}$ and $b = \frac{1}{2}x^{-\frac{4}{9}}y^{\frac{1}{9}}$.

29. Simplify $\frac{1}{8^{-\frac{2}{3}}} - 3x^0 + 27^{-\frac{1}{3}} - 1^{\frac{1}{2}}$.

30. Solve graphically

$$x^2 + y = 7.$$

$$x + y^2 = 11.$$

31. Given $y^2 - 4xy + 3x^2 + 6y - 4x + 4 = 0$. Solve for y in terms of x .

32. Express in the form $a + bi$: $\frac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}}$.

33. Find the value of x in the following inequalities:

(a) $5x > \frac{3x}{2} + 14$. (c) $ax - b > cx + d$.

(b) $\frac{5x}{8} + \frac{5}{4} < \frac{11}{6} + \frac{7x}{12}$. (d) $\frac{x-a}{b} < 1 - \frac{x}{a}$.

34. Solve $x^2 - x - 20 < 0$.

35. Given $5x + 3y > 121$ and $7x + 4y = 168$.

Find the limits of x and y .

36. If $3x - 2 > \frac{1}{2}x - \frac{4}{5}$, and $\frac{7}{6} - \frac{5}{4}x < 8 - 2x$, show that the values of x lie between $\frac{12}{5}$ and $\frac{82}{9}$.

37. In a problem called "The Problem of the Couriers" we obtain the equations $x = \frac{a}{m-n}$ and $y = \frac{am}{m-n}$.

Assume $m > 0$, $n > 0$, $a > 0$.

Discuss these values:

(a) If $m > n$. (d) If $a = 0$ and $m > n$ or $m < n$.

(b) If $m < n$. (e) If $a = 0$ and $m = n$.

(c) If $m = n$ or $m - n = 0$.

38. When $x \rightarrow a$, what is the limiting value of

$$\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt[3]{x}}?$$

39. If $f(x) = x^3 - 10x^2 + 31x - 30$, find $f(2)$, $f(3)$.

40. Find the limit of $\left[\frac{x^2 - 2x + 5}{x^2 + 7} \right]_{x \rightarrow 1}$.

41. What is the limit of $\left[\frac{x^3 + a^3}{x^2 - a^2} \right]_{x \rightarrow -a}$?

42. Find $\lim_{x \rightarrow 0} \left[\frac{8x^2 + 2x}{2x^2 + 4x} \right]$.

43. In how many ways can 2 prizes be awarded to a class of 10 boys, if both prizes may be given to the same boy?

44. How many different numbers of seven figures can be formed from the digits 1, 1, 2, 3, 3, 3, 4?

45. How many diagonals can be drawn in a polygon of 100 sides?

46. If ${}_m C_{m-1} = {}_{n+1} C_{n-1}$, prove that $n(n+1) = 2m$.

47. The number of permutations of n things taken 5 at a time is equal to 120 times the number of combinations of the n things taken 3 at a time. Find n .

48. The number of combinations of n different things taken r together was found to be twice the number taken $r-1$ together and also to be four fifths the number taken $r+1$ together. Find n and r .

49. How many different sums of money can be made up from a cent, a nickel, a dime, and a quarter?

50. Find the number of distinct permutations of the letters of the word "committee," using all the letters in each permutation.

51. Deduce the formula for the number of permutations of n things taken all at a time, when p are of one kind, q of another kind, and r of a third kind.

52. State accurately any theorem that you have used in problems in chance (probability). In a certain club there are 25 Democrats and 15 Republicans. If a committee of 4 is chosen by lot, what is the chance that all will be Republicans?

53. In a certain school there are 60 pupils and 5 teachers. An athletic committee of 3 teachers and 2 pupils is to be chosen. How many such committees could be formed?

54. Find the coefficient of x^4 in $\left(2x^2 - \frac{1}{4x}\right)^8$.

55. Show that $(a + b)^4$ will not be equal to $a^4 + b^4$, unless $ab = 2(a + b)^2$ (assuming that neither a nor b is zero).

56. Write and simplify the first three terms of

$$(x^{\frac{2}{3}} - 2x^{\frac{1}{3}})^{10}.$$

57. Find the term of the expansion of $\left(x^{\frac{1}{2}} - \sqrt{\frac{y}{x}}\right)^{10}$ which does not contain x .

58. Find two terms of $(1 + x)^{15}$ which will have the greatest coefficient.

59. Determine for what values of k the roots of the quadratic equation $kx^2 + 2kx - 3x + 2 = 0$ are real and equal, and check your results.

60. Show that $3ax^2 - (2b + 3c)x + 2c = 0$ has rational roots.

61. Form an equation whose roots shall be the cubes of the roots of the equation $2x(x - a) = a^2$.

62. For what values of k and l will one of the roots of the equation $kx^2 + lx + k$ be the reciprocal of the other?

63. By synthetic division, divide

$$x^3 - 2x^2 - 4x + 8 \text{ by } (x - 3).$$

64. One root of $x^3 - 6x + 9 = 0$ is $\frac{1}{2}(3 + \sqrt{-3})$. Find the others.

65. What is the equation of which the roots are $1 + \sqrt{-5}$, $1 - \sqrt{-5}$, $+\sqrt{5}$, and $-\sqrt{5}$?

66. Solve by trial $6x^3 - 41x^2 + 31x - 6 = 0$.

67. Transform $6x^3 + 23x^2 - 6x - 8 = 0$ into an equation with integral coefficients and first coefficient unity. Solve both equations and explain the relation between the roots.

68. Find out all you can about the rational roots of the equation

$$x^4 - 5x^3 + 20x - 16 = 0.$$

69. A root of the following equation lies between -3 and -4 . Find its value correct to three significant figures.

$$x^3 + x^2 - 4x + 16 = 0.$$

70. (a) Graph $y = x^3 + 2x^2 - 5x - 9$.

(b) Find to two decimal places the positive root of

$$x^3 + 2x^2 - 5x - 9 = 0.$$

71. Solve by Cardan's method $x^3 + 6x = 88$.

72. State without proof two theorems which you have found most useful in evaluating determinants.

Show that

$$\begin{vmatrix} x & p & q & 1 \\ a & x & r & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = (x - a)(x - b)(x - c).$$

73. Find the value of x from the following simultaneous equations, by determinants:

$$x - 2y + z = 1.$$

$$2y + 3z = 2.$$

$$2x + y - 4z = -3.$$

74. Evaluate the determinant:

$$\begin{vmatrix} 2 & 2 & 4 & 2 \\ 1 & 4 & 1 & 4 \\ 1 & 1 & 2 & 2 \\ 4 & 8 & 13 & 11 \end{vmatrix}.$$

75. Find the values of x that reduces the following determinant to 0:

$$\begin{vmatrix} 5 & 1 & x \\ 8 & x & 2 \\ 4 & x & 4 \end{vmatrix}.$$

76. (a) Prove that if two rows in a determinant are identical, the value of the determinant is zero.

(b) Solve for x ,

$$\begin{vmatrix} 3x & 2 & 1 & 0 \\ 2 & 0 & 1 & 3 \\ x & 5 & 0 & 1 \\ -x & 7 & 2 & 0 \end{vmatrix} = 0.$$

77. Resolve $\frac{1 - x + 2x^2}{(1 - x)^3}$ into partial fractions and determine the coefficient of x^3 , if the expression be expanded in ascending powers of x .

78. Resolve into partial fractions $\frac{x}{x^2 - 2x - 3}$.

79. Expand by the Binomial Theorem $(\frac{1}{2}x - y^{-1})^{-2}$ to 4 terms.

80. Given $729^{\frac{1}{x}} = 3$, find the value of x .

81. Find the value of $\left[\frac{2.186\sqrt{0.965}}{38.23} \right]^3$.

82. Prove and state the reason for each step, that

$$\log \left\{ \frac{\sqrt{1 + x^2} - 1}{\sqrt{1 + x^2} + 1} \right\} = 2 \{ \log (\sqrt{1 + x^2} - 1) - \log x \}.$$

83. Solve the equation $3^x = 226$ by logarithms.

84. Find the value of

$$\log_3 9 + \log_{21} \frac{1}{8} + \log_{10} 10^{\frac{1}{4}}.$$

85. If $\log a = 8.9232 - 10$ and $\log b = 8.5702 - 10$, find a and b , and then find $\log \sqrt[3]{a + b}$.

86. Find by logarithms the value of $\sqrt[3]{\frac{0.6712}{5.327}}$.

87. Find the present value of \$2000 payable in 15 years from now, money being worth 4%.

Hint: $P. V. = 2000(1 + r)^{-15}$.

88. A man sells a piece of property for \$8000 to be paid as follows: \$1000 cash, and the balance in seven annual installments of \$1000 each. Find the equivalent total cash payment, if money is worth 6%.

89. An insurance premium of \$150 is paid each year for 20 years. Find the value of the sum of these premiums at the end of the twentieth year, with interest at 4% compounded annually, if

$$\text{Value} = \text{Premium} \times R \left(\frac{R^n - 1}{r} \right),$$

where $R = 100\% + \text{the } \%$, $r = \text{the rate } \%$, and $n = \text{number of years}$.

90. Find the maximum or minimum point of the curve represented by the equation $y = -\frac{x^2}{4} + x - 1$.

91. An open box is to be made out of a square piece of cardboard 6 inches on a side, by cutting out equal squares from the corners and then turning up the sides. If x represents a side of the square cut out from each corner, (a) express the volume y of the box in terms of x , (b) make a graph of this relation between y and x , (c) from the graph estimate the value of x that will give the largest box.

92. Find the positive integral solutions of the indeterminate equation $7x + 12y = 220$.

93. Brown sold chickens and turkeys, receiving \$16 for them. If he received 75 cents for each chicken and \$2 for each turkey, how many of each did he sell?

94. Find the value of $\frac{1110}{13}$, the number being written in the scale of four. Perform the indicated division in the scale of four, then change the numbers to the scale of ten and check your results.

95. Find the value of $\frac{21 \times 143}{34}$, the numbers being written in the scale of seven. Perform the indicated multiplications and division in the scale of seven, then change to the scale of ten and check your results.

The following miscellaneous exercises are taken from a recent examination for mathematics teachers in N. Y. City.

96. In the equation x^3 plus $2x$ minus 20 equals 0, find the root between 2 and 3.

97. Two of the roots of $2x^3$ minus $7x^2$ plus $10x$ minus 6 equals 0 are $1 + i$ and $1 - i$. Find the third root.

98. At his usual rate a man can row 15 miles downstream in 5 hours less time than it takes him to return. Could he double his rate, his time downstream would be only 1 hour less than his time upstream. What is his usual rate in dead water and what is the rate of the current?

99. Three men, A, B, and C, set out at the same time to walk a certain distance. A walks $4\frac{1}{2}$ miles an hour and finishes the journey 2 hours before B. B walks 1 mile an hour faster than C and finishes the journey in 3 hours' less time. What is the distance?

100. Solve the following system of equations by determinants:

$$2x \text{ plus } 3y \text{ minus } 5z \text{ equals } 3.$$

$$x \text{ minus } 2y \text{ plus } z \text{ equals } 0.$$

$$3x \text{ plus } y \text{ plus } 3z \text{ equals } 7.$$

101. Evaluate the following determinant:

$$\begin{vmatrix} 6 & 42 & 27 \\ 8 & -28 & 36 \\ 20 & 35 & 135 \end{vmatrix}.$$

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